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Quantum noise and stochastic reduction

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Abstract
In standard nonrelativistic quantum mechanics the expectation of the energy is a conserved quantity. It is possible to extend the dynamical law associated with the evolution of a quantum state consistently to include a nonlinear stochastic component, while respecting the conservation law. According to the dynamics thus obtained, referred to as the energy-based stochastic Schrödinger equation, an arbitrary initial state collapses spontaneously to one of the energy eigenstates, thus describing the phenomenon of quantum state reduction. In this paper, two such models are investigated: one that achieves state reduction in infinite time and the other in finite time. The properties of the associated energy expectation process and the energy variance process are worked out in detail. By use of a novel application of a nonlinear filtering method, closed-form solutions—algebraic in character and involving no integration—are obtained of both these models. In each case, the solution is expressed in terms of a random variable representing the terminal energy of the system and an independent noise process. With these solutions at hand it is possible to simulate explicitly the dynamics of the quantum states of complicated physical systems.

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1. Introduction

The idea that the standard Schrödinger equation of nonrelativistic quantum mechanics should be extended to take the form of a stochastic differential equation on Hilbert space has been investigated extensively as a mathematically viable approach to the measurement problem in quantum mechanics. Indeed, there is now a substantial body of literature on the theory of spontaneous state-vector reduction, and a number of different models have been proposed that fall into this category. See, e.g., [3, 7, 24, 27] for overviews of this area and relevant references.

This paper is concerned with the so-called energy-based stochastic extension of the Schrödinger equation, which has the special status of being the simplest such extension that is in principle applicable to any nonrelativistic quantum system. The physical set-up can be described briefly as follows. We consider an isolated quantum system for which the Hamiltonian $\hat{H}$ has a discrete spectrum $\{E_i\}_{i=1,2,\ldots,N}$. We assume that initially the system is in a pure state represented by the state vector $|\psi_0\rangle$. The situation where the initial state is mixed can also be considered (see, e.g., [4]), but for simplicity we confine the discussion to the case of an initially pure state in this paper. For convenience, we set $\langle\psi_0|\psi_0\rangle = 1$. For each value of i we let $|\phi_i\rangle$ denote the Lüders state associated with the energy level E_i. More specifically, let us write $\hat{\Pi}_i$ for the projection operator onto the Hilbert subspace of states for which the energy has the value E_i. We allow for the possibility that the energy may be degenerate. In that case, we write n_i for the number of linearly independent state vectors with energy E_i. Then the Hamiltonian takes the form $\hat{H} = \sum_i E_i\hat{\Pi}_i$, and we define the Lüders states by setting

$$|\phi_i\rangle = \frac{\hat{\Pi}_i|\psi_0\rangle}{\langle\psi_0|\hat{\Pi}_i|\psi_0\rangle^{1/2}}. \tag{1}$$

We note that $\hat{H}|\phi_i\rangle = E_i|\phi_i\rangle$ and $\langle\phi_i|\phi_i\rangle = 1$. According to the von Neumann–Lüders projection postulate [22, 30], if the system is initially in the pure state $|\psi_0\rangle$ and if the outcome of a measurement of the energy is the eigenvalue E_i, then after the measurement the state of the system will be the Lüders state $|\phi_i\rangle$.

It is an implicit feature of the projection postulate that quantum evolution progresses in accordance with the unitary dynamics of the Schrödinger equation up to the moment when a measurement is made, at which point the system jumps to a new state. The von Neumann–Lüders rule asserting how the jump proceeds is essentially ad hoc in nature, despite being plausible from a physical point of view insofar as the predicted outcome is concerned. Thus although the projection postulate, in one form or another, remains an accepted part of the everyday use of quantum theory in practical applications [19], one has to agree that such a 'cookbook' approach to the measurement problem is ultimately unsatisfactory; and this is why over the last five decades many attempts have been made to modify the dynamics of standard quantum mechanics in such a way that the 'collapse' process can be understood as governed by an evolutionary law that operates on a universal basis, rather as does the Schrödinger equation in ordinary quantum mechanics.

In order to ensure consistency with established facts, such a universal evolutionary law needs to have the property that for some systems it proceeds in a way that for all practical purposes reproduces the dynamics of the Schrödinger equation, whereas for other systems the evolution progresses continuously to a terminal state that is consistent with the action of the projection postulate. This 'viability' property is satisfied in particular by the standard energy-based stochastic Schrödinger equation. In this model the Schrödinger equation, which when written in differential form is given by

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}|\psi_t\rangle\,\mathrm{d}t, \tag{2}$$

is generalized and elevated to the status of a nonlinear stochastic differential equation on Hilbert space:

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}|\psi_t\rangle\,\mathrm{d}t - \tfrac{1}{8}\sigma^2(\hat{H} - H_t)^2|\psi_t\rangle\,\mathrm{d}t + \tfrac{1}{2}\sigma(\hat{H} - H_t)|\psi_t\rangle\,\mathrm{d}W_t. \tag{3}$$

Here, $\{|\psi_t\rangle\}_{0\leqslant t<\infty}$ is the state-vector process, $\{W_t\}_{0\leqslant t<\infty}$ is the Wiener process and $\{H_t\}_{0\leqslant t<\infty}$ is the energy expectation process, defined by

$$H_t = \frac{\langle\psi_t|\hat{H}|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle}. \tag{4}$$

The coupling constant σ appearing in (3), which has the units $\sigma \sim [\text{energy}]^{-1}\,[\text{time}]^{-1/2}$, determines the characteristic timescale τ_R associated with the rate of collapse of the wavefunction induced by (3). This timescale is given typically by an expression of the form $\tau_R = 1/\sigma^2 V_0$, where V_0 is the initial value of the process $\{V_t\}_{0\leqslant t<\infty}$ for the energy variance, which at time t is given by the following expression:

$$V_t = \frac{\langle\psi_t|(\hat{H} - H_t)^2|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle}. \tag{5}$$

One of the attractive features of the stochastic differential equation (3) is that it provides a more or less completely tractable model for state-vector reduction in nonrelativistic quantum mechanics. Nevertheless, despite the fact that the mathematical properties and physical consequences of (3) and various related processes have been studied extensively in the literature [1, 4, 5, 10, 15–18, 26], it has only been recently that a general solution of (3) has been obtained in terms of an appropriate set of freely specifiable random data [11, 12]. The aim of this paper is to present a complete treatment of the method of solution of the dynamical equation (3). The results are of interest both for the new range of numerical and computational techniques they open up, as well as for the new methods for model building they provide.

The paper is organized as follows. In section 2, we review some basic notions of stochastic analysis, including the concepts of filtrations, conditional expectations, martingales, supermartingales and potentials. The material introduced in this section will be used throughout the paper. In sections 3–5, we establish various properties of the energy expectation process (4) and the energy variance process (5), showing that the variance process has the 'potential' property; that is to say, its expectation goes to zero asymptotically. This allows us to give a precise sense to the notion of state reduction. In section 6, we then determine the circumstances under which the state vector reduces to one of the eigenstates of an observable that is compatible with the Hamiltonian.

In section 7, we address the problem of the origin of the dynamical equation (3). Starting from a general stochastic equation for a state vector driven by a single Brownian motion, we determine what additional physical assumptions and other simplifying features are required in order to obtain (3). We also show, under a suitable 'universality' assumption regarding energy conservation, that reduction to lower energy uncertainty is a generic feature of the stochastic Schrödinger equation.

The projection operators for the energy eigenstates constitute a special set of observables that commute with the Hamiltonian. The expectation value, with respect to the state $|\psi_t\rangle$, of such a projection operator determines the random process for the associated conditional transition probability to that eigenstate. The properties of this conditional probability process are studied in section 8. In section 9, we digress briefly to investigate the dynamics of the Shannon entropy associated with the system of transition probabilities and show that the Shannon entropy has the property that its expectation goes to zero asymptotically. This is contrasted with the behaviour of the von Neumann entropy. We also derive an equality that relates the entropic measure and the variance-based measure of energy dispersion. In section 10, we study a certain linear stochastic differential equation for the state vector, which we call the ancillary equation, and verify that a state vector satisfying the ancillary equation, once suitably normalized, can be used as a step to obtaining the solution of the nonlinear equation (3). We also clarify the relation of our results to earlier work on solutions of (3) and related dynamical equations, explaining why previously established integral representations for the state vector satisfying the stochastic equation should not generally be regarded as explicit solutions in the sense that we use the term here.

Then in sections 11 and 12, we derive a bona fide explicit solution to (3), making use of a nonlinear filtering method. The solution thus obtained is expressed in terms of a simple algebraic function of a standard Brownian motion and an independent random variable representing the terminal value of the energy. By use of this result, it is possible to simulate solutions that represent the evolution of rather complicated quantum systems. In section 13, we introduce a technique that allows one to verify that the solution obtained in section 11 does indeed give rise to the reduction of the state vector. In section 14, we investigate properties of the asymptotic random variable corresponding to the terminal value of the energy. While in the nonlinear filtering method used to solve (3) we introduce a noise term, in section 15 we derive the external noise term from the underlying processes specified in (3). This result justifies the use of the filtering methodology we have employed here.

We then turn to solve the problem of constructing a collapse model that achieves state reduction in finite time. That is, although the standard energy-based collapse model (3) achieves a strict collapse in infinite time, with a minimal modification of the dynamical equation (3) it is possible to formulate a finite-time collapse model. In section 16, we introduce such a model. Using the methodology of section 11, we derive an analytical expression that we conjecture to give the energy expectation process. The validity of this conjecture is established in section 17. In section 18, we derive the external noise term arising in the finite-time collapse model that is used in section 17 to obtain the solution. In section 19, we demonstrate the fact that the standard energy-based model (3) and the finite-time collapse model introduced in (198) are related by a nonlinear time-change. That is, if we take the model (3) and replace the time variable t by a 'clock' variable defined by $\tau(t) = tT/(T-t)$, where T is a finite positive constant, then in a physical world measured by the variable t, the collapse for the new system takes place in finite time interval T, since the clock variable $\tau(t)$ runs from 0 to ∞ as t runs from 0 to T.

As a closing remark, in section 20, the role of the asymptotic value of the energy, which has the interpretation of a hidden variable in the stochastic framework, is discussed. We also speculate on whether the energy-based reduction models analysed here suffice as such to form a basis for the general description of random phenomena in nonrelativistic quantum mechanics.

2. Stochastic essentials

We begin with an overview of the probabilistic framework implicit in the specification of the energy-based stochastic Schrödinger equation. The concepts introduced in this section are standard in the literature of stochastic analysis, as is also the notation (see, e.g., [20, 21, 28, 29, 32, 33]). The dynamics of the state vector $|\psi_t\rangle$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{0\leqslant t<\infty}$, with respect to which the process $\{W_t\}_{0\leqslant t<\infty}$ is a standard Brownian motion. Here, Ω is the sample space on which $\mathcal{F}$ is a σ-algebra of open sets upon which the probability measure $\mathbb{P}$ is defined. Each element $\omega \in \Omega$ represents a 'possible outcome of chance'. Each element $A \in \mathcal{F}$ is an 'event'. The measure $\mathbb{P}$ assigns a probability $\mathbb{P}(A)$ to each event A.

Now we give the relevant definitions in more detail, since these are of interest. Let Ω be a set, and let $\mathcal{F}$ be a collection of subsets of Ω. For any subset $A \subset \Omega$ we let $A^{\mathrm{c}} = \{\omega \in \Omega | \omega \notin A\}$ denote its complement. Then $\mathcal{F}$ is called an algebra of subsets of Ω if (a) $\Omega \in \mathcal{F}$, (b) $A \in \mathcal{F}$ implies that $A^{\mathrm{c}} \in \mathcal{F}$ and (c) $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$. It follows from these axioms that $\emptyset \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

The algebraic operations on the elements of $\mathcal{F}$ are as follows. The product of two elements $A, B \in \mathcal{F}$ is defined by $A \cdot B = A \cap B$, and the sum of two elements $A, B \in \mathcal{F}$ is defined by $A + B = (A \cup B) \cap (A \cap B)^{\mathrm{c}}$. It follows that the product and sum operations are symmetric and associative and that $(A + B) \cdot C = A \cdot C + B \cdot C$ for any $A, B, C \in \mathcal{F}$. The underlying field of the algebra $\mathcal{F}$ is the minimal subalgebra $\{\Omega, \emptyset\}$, which when endowed with the same product and sum operations as those defined above satisfies the rules of binary arithmetic:

$$\begin{cases} \emptyset \cdot \emptyset = \emptyset, & \emptyset \cdot \Omega = \emptyset, \qquad \Omega \cdot \Omega = \Omega, \\ \emptyset + \emptyset = \emptyset, & \emptyset + \Omega = \Omega, \qquad \Omega + \Omega = \emptyset. \end{cases} \tag{6}$$

If $\mathcal{F}$ is an algebra of subsets of Ω then we say that $\mathcal{F}$ is a σ-algebra if it has the property that whenever $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}$ then $\cup_n A_n \in \mathcal{F}$. That is to say, the union of any countable sequence of elements of $\mathcal{F}$ is also an element of $\mathcal{F}$. It follows that whenever $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}$ we have $\cap_n A_n \in \mathcal{F}$, since $\cap_n A_n = \left(\cup_n A_n^{\mathrm{c}}\right)^{\mathrm{c}}$.

We comment briefly on the distinction between a σ-algebra and a topology, since the latter is more familiar to physicists than the former. In a topology we axiomatize the notion of an open set and require that the union of any collection of open sets is open and that the intersection of any finite collection of open sets is open. In a σ-algebra we axiomatize the notion of a measurable set and require that the union of any countable sequence of measurable sets is measurable and that the intersection of any countable sequence of measurable sets is measurable.

If $\mathcal{F}$ is a σ-algebra of subsets of a set Ω, then we call the pair $(\Omega, \mathcal{F})$ a measurable space. If $(\Omega, \mathcal{F})$ is a measurable space, then a probability measure on $(\Omega, \mathcal{F})$ is a map $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying: (a) $\mathbb{P}(\emptyset) = 0$, (b) $\mathbb{P}(\Omega) = 1$ and (c) if $\{A_n\}_{n\in\mathbb{N}}$ is a countable sequence of disjoint elements of $\mathcal{F}$ with union $A = \cup_n A_n$, then $\mathbb{P}(A) = \sum_n \mathbb{P}(A_n)$. A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

The introduction of the concept of a filtration on a probability space allows one to formalize the notion that the consequences of the outcome of chance are not necessarily revealed at once, but rather may emerge sequentially as time progresses. More specifically, a filtration of $\mathcal{F}$ is a collection $\{\mathcal{F}_t\}$ of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all s and t such that $0 \leqslant s \leqslant t < \infty$.

If an event $A \in \mathcal{F}$ is such that $A \in \mathcal{F}_t$ for some given value of t, then we interpret this to mean that at time t one can say whether $\omega \in A$ or not. To put this another way, in a filtered probability space each $\omega \in \Omega$ corresponds to a possible 'future history'. Each element $A \in \mathcal{F}_t$ then represents a simple yes/no question, the answer to which, for any particular future history, will be known for certain by time t. For that reason, the nesting $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leqslant t$ gives rise to a notion of causality.

A real-valued function $X : \Omega \mapsto \mathbb{R}$ is said to be *measurable* with respect to the σ-algebra $\mathcal{F}$ if for each number $x \in \mathbb{R}$ the set $\{\omega \in \Omega : \mathbf{1}_x(\omega) = 1\}$ is an element of $\mathcal{F}$. Here, $\mathbf{1}_x(\omega)$ is the indicator function on Ω for the set consisting of all ω such that $X(\omega) \leqslant x$. Thus, $\mathbf{1}_x : \Omega \mapsto \{0, 1\}$ and $\mathbf{1}_x(\omega) = \mathbf{1}_{\{X(\omega)\leqslant x\}}$. If X is $\mathcal{F}$-measurable in the sense just discussed, we say that X is a real-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. The probability distribution function $F_X(x) = \mathbb{P}(X \leqslant x)$ is then defined by use of the Lebesgue integral:

$$\mathbb{P}(X \leqslant x) = \int_\Omega \mathbf{1}_x(\omega)\, \mathrm{d}\mathbb{P}(\omega). \tag{7}$$

More generally, we also consider maps of the form $X : \Omega \mapsto \Gamma$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(\Gamma, \mathcal{G})$ is a measurable space. For example, Γ could be $\mathbb{R}^n$, $\mathbb{C}^n$, a Hilbert space or a manifold. In that case we say the random variable X takes values in Γ and $\mathcal{G}$ can be typically taken to be the so-called Borel σ-algebra generated by the open sets of Γ. Then for any element $G \in \mathcal{G}$, we define

$$\mathbb{P}(X \in G) = \int_\Omega \mathbf{1}_{\{X(\omega)\in G\}}\, \mathrm{d}\mathbb{P}(\omega). \tag{8}$$

A parametric family $\{X_t\}_{0\leqslant t<\infty}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a random process. If a random process $\{X_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$ has the property that for each value of t the random variable X_t is $\mathcal{F}_t$-measurable, then we say that $\{X_t\}$ is *adapted* to the filtration $\{\mathcal{F}_t\}$.

If X is a nonnegative real random variable, its $\mathbb{P}$-expectation (i.e., its expectation with respect to the measure $\mathbb{P}$) is defined by the integral

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega), \tag{9}$$

which may take the value $+\infty$. More generally, when X is not necessarily nonnegative, the expectation is defined only when one of the expressions $\mathbb{E}[X^+]$ or $\mathbb{E}[X^-]$ is finite, where $X^+ = \max(X, 0)$ and $X^- = -\min(X, 0)$, in which case $\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$. A random variable such that $\mathbb{E}[|X|] = \mathbb{E}[X^+] + \mathbb{E}[X^-]$ is finite is said to be *integrable*.

Now we turn to the definition of conditional expectation. Given a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ for which $\mathbb{E}[X]$ exists, the conditional expectation $\mathbb{E}[X|\mathcal{A}]$ of X with respect to the σ-subalgebra $\mathcal{A} \subset \mathcal{F}$ is defined to be any $\mathcal{A}$-measurable random variable Y for which $\mathbb{E}[Y]$ is defined, such that for any element $A \in \mathcal{A}$ we have

$$\int_\Omega \mathbf{1}_A(\omega) X(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \mathbf{1}_A(\omega) Y(\omega)\, \mathrm{d}\mathbb{P}(\omega). \tag{10}$$

If such a random variable exists, then it is unique up to equivalence modulo differences on sets of $\mathbb{P}$-measure zero. Thus even if $\mathbb{E}[X|\mathcal{A}]$ is not quite unique we refer to it as *the* conditional expectation of X with respect to $\mathcal{A}$. This definition, which at first glance appears rather formal and indirect, is nevertheless one of the cornerstones of modern probability theory and is indispensable. We remark that a sufficient condition for $\mathbb{E}[X|\mathcal{A}]$ to exist is that X should be integrable.

The following properties of the conditional expectation are often useful in calculations: (i) the law of total probability $\mathbb{E}[\mathbb{E}[X|\mathcal{A}]] = \mathbb{E}[X]$ and (ii) the tower property, which says that if $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$ then $\mathbb{E}[\mathbb{E}[X|\mathcal{B}]|\mathcal{A}] = \mathbb{E}[X|\mathcal{A}]$. The law of total probability is a special case of the tower property.

The conditional expectation operation allows us to introduce the concept of a *martingale*, the stochastic analogue of a conserved quantity. For this purpose, we need the operation of conditioning with respect to a σ-subalgebra $\mathcal{F}_t$ belonging to a filtration $\{\mathcal{F}_t\}_{0\leqslant t<\infty}$. Intuitively, conditioning with respect to $\mathcal{F}_t$ means conditioning with respect to the information that will become available up to time t. For convenience, we often use the abbreviation $\mathbb{E}_t[X] = \mathbb{E}[X|\mathcal{F}_t]$ when the choice of filtration can be taken as understood. There are situations, however, where more than one filtration may arise in the context of a given problem, in which case the more explicit notation is useful. The conditional expectation $\mathbb{E}_t[X]$ satisfies $\mathbb{E}[\mathbb{E}_t[X]] = \mathbb{E}[X]$ and $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_s[X]$ for $s \leqslant t$. We note that if X is $\mathcal{F}_t$-measurable, then $\mathbb{E}_t[X] = X$.

A real-valued process $\{X_t\}$ is said to be an $\{\mathcal{F}_t\}$-*martingale* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if $\mathbb{E}[|X_t|] < \infty$ for all $0 \leqslant t < \infty$ and $\mathbb{E}_s[X_t] = X_s$ for all $0 \leqslant s \leqslant t < \infty$. In other words, $\{X_t\}$ is an $\{\mathcal{F}_t\}$-martingale if it is integrable and if for $t \geqslant s$ the conditional expectation of X_t, given $\mathcal{F}_s$, is the value X_s of the process at time s. A process $\{X_t\}$ is an $\{\mathcal{F}_t\}$-*supermartingale* on $(\Omega, \mathcal{F}, \mathbb{P})$ if $\mathbb{E}[|X_t|] < \infty$ for all $t \geqslant 0$, and $\mathbb{E}_s[X_t] \leqslant X_s$ for all $0 \leqslant s \leqslant t < \infty$. Intuitively, a supermartingale is a process that tends on average, at any time, to be nonincreasing. A martingale is *a fortiori* a supermartingale. The martingale convergence theorem (see, e.g., [28], theorem 10) states that if $\{X_t\}$ is a supermartingale that satisfies

$$\sup_{0\leqslant t<\infty} \mathbb{E}[|X_t|] < \infty, \tag{11}$$

then there exists a random variable Y such that $\lim_{t\to\infty} X_t = Y$ almost surely (i.e., with probability one) and that $\mathbb{E}[|Y|] < \infty$. It follows that a positive supermartingale necessarily converges to a limit as t goes to infinity. A positive supermartingale $\{X_t\}$ with the property that $\lim_{t\to\infty} \mathbb{E}[X_t] = 0$ is called a *potential* [23].

We now review some basic formulae arising in the theory of Ito processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{0\leqslant t<\infty}$, and let $\{W_t\}_{0\leqslant t<\infty}$ be a standard Wiener process adapted to $\{\mathcal{F}_t\}$. Here by a standard Wiener process (or Brownian motion) we mean a continuous process $\{W_t\}$ with the properties that (i) $\{W_t\}$ has independent increments, and that (ii) $W_t - W_s$ for $0 \leqslant s < t < \infty$ is a Gaussian random variable with mean zero and variance $t - s$.

Let $\{a_t\}_{0\leqslant t<\infty}$ and $\{b_t\}_{0\leqslant t<\infty}$ be $\{\mathcal{F}_t\}$-adapted processes such that for any $t \in [0, \infty)$ we have

$$\int_0^t |a_s|\,\mathrm{d}s + \int_0^t b_s^2\,\mathrm{d}s < \infty \tag{12}$$

almost surely. Then letting X_0 an $\mathcal{F}_0$-measurable initial condition, the random variable X_t defined by the stochastic integral

$$X_t = X_0 + \int_0^t a_s\,\mathrm{d}s + \int_0^t b_s\,\mathrm{d}W_s \tag{13}$$

is well defined and $\mathcal{F}_t$-measurable for all t, and we call $\{X_t\}_{0\leqslant t<\infty}$ an Ito process (see, e.g., [28, 29] for the general definition of the stochastic integral). In this case, we say that $\{X_t\}$ is a real-valued Ito process driven by the one-dimensional Wiener process $\{W_t\}$. It is straightforward to generalize (13) to cases for which both $\{X_t\}$ and $\{W_t\}$ are multidimensional.

One useful tool of which we make repeated use is Ito's lemma. Suppose $\{X_t\}$ is given by (13) and consider the process $\{f_t\}_{0\leqslant t<\infty}$ defined by $f_t = f(X_t, t)$ where $f \in C^{2,1}(\mathbb{R}\times\mathbb{R}^+)$. Let prime and dot denote differentiation with respect to the first and second arguments of $f(x, t)$, respectively. Then Ito's lemma states that

$$\begin{aligned} f(X_t, t) = f(X_0, 0) &+ \int_0^t \dot{f}(X_s, s)\,\mathrm{d}s + \int_0^t a_s f'(X_s, s)\,\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t b_s^2 f''(X_s, s)\,\mathrm{d}s + \int_0^t b_s f'(X_s, s)\,\mathrm{d}W_s. \end{aligned} \tag{14}$$

It is often convenient to express (13) and (14) in differential form: thus we write

$$\mathrm{d}X_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}W_t \tag{15}$$

for the 'dynamics' of $\{X_t\}$, and

$$\mathrm{d}f(X_t, t) = \left(\dot{f}(X_t, t) + a_t f'(X_t, t) + \tfrac{1}{2} b_t^2 f''(X_t, t)\right)\mathrm{d}t + b_t f'(X_t, t)\,\mathrm{d}W_t \tag{16}$$

for the dynamics of $\{f_t\}$ implied by Ito's lemma. As in ordinary calculus, the differential equations of stochastic calculus are essentially formal in character and always derive their meaning from associated integral equations. Thus, (15) and (16) refer back to (13) and (14). Nevertheless, as in ordinary calculus, the manipulation of infinitesimal quantities in stochastic calculus can be very powerful as a mathematical technique and can be intuitively very suggestive as well. For example, the so-called Ito product rule

$$\mathrm{d}(X_t Y_t) = Y_t\,\mathrm{d}X_t + X_t\,\mathrm{d}Y_t + \mathrm{d}X_t\,\mathrm{d}Y_t \tag{17}$$

is shorthand for the fact that if $\{X_t\}$ is given by (13) and $\{Y_t\}$ is given analogously, but with $\{a_t\}$ and $\{b_t\}$ replaced by $\{p_t\}$ and $\{q_t\}$, then

$$X_t Y_t = X_0 Y_0 + \int_0^t (Y_s a_s + X_s p_s + b_s q_s)\,\mathrm{d}s + \int_0^t (Y_s b_s + X_s q_s)\,\mathrm{d}W_s. \tag{18}$$

Now consider an Ito process $\{M_t\}_{0\leqslant t<\infty}$ of the form

$$M_t = M_0 + \int_0^t b_s \, \mathrm{d}W_s. \tag{19}$$

Then a sufficient condition for $\{M_t\}$ to be a martingale is that M_0 should be integrable, and that

$$\mathbb{E}\left[\int_0^t b_s^2 \, \mathrm{d}s\right] < \infty \tag{20}$$

for all $t \in [0, \infty)$. In that case $\{M_t\}$ is called a square-integrable martingale, and we have the identity

$$\mathbb{E}[(M_t - M_0)^2] = \mathbb{E}\left[\int_0^t b_s^2 \, \mathrm{d}s\right]. \tag{21}$$

More generally, we also have the following relation, valid for $t \geqslant s \geqslant 0$, which we call the *conditional Wiener–Ito isometry*:

$$\mathbb{E}_s[(M_t - M_s)^2] = \mathbb{E}_s\left[\int_s^t b_u^2 \, \mathrm{d}u\right]. \tag{22}$$

In certain situations we are presented with an equation of the form (13), and we are told the distribution of X_0 and that the processes $\{a_t\}$ and $\{b_t\}$ are of the form $a_t = a(X_t, t)$ and $b_t = b(X_t, t)$, where $a(x, t)$ and $b(x, t)$ are prescribed functions. In that case we have a *stochastic differential equation* of the form

$$\mathrm{d}X_t = a(X_t, t) \, \mathrm{d}t + b(X_t, t) \, \mathrm{d}W_t, \tag{23}$$

with initial condition X_0. By a 'solution' of the stochastic differential equation (23) we mean the specification of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$, together with an $\{\mathcal{F}_t\}$-adapted Brownian motion and an $\{\mathcal{F}_t\}$-adapted Ito process $\{X_t\}$ satisfying (23) along with the given initial condition.

The extension of these definitions to situations where $\{X_t\}$ and $\{W_t\}$ are multidimensional is straightforward. It is also appropriate in some circumstances to consider processes defined over a finite time horizon $t \in [0, T]$, $T < \infty$, for which straightforward modifications of the relevant definitions can also be formulated.

3. Dynamics of the energy process

Now we are in a position to analyse the dynamics of the energy-based stochastic Schrödinger equation (3) in more detail. We shall make the following assumptions concerning the dynamics of the state vector:

(a) The state-vector process $\{|\psi_t\rangle\}_{0\leqslant t<\infty}$ takes values in a finite-dimensional complex Hilbert space and is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{0\leqslant t<\infty}$.
(b) $\{|\psi_t\rangle\}$ is adapted to $\{\mathcal{F}_t\}$.
(c) $\{|\psi_t\rangle\}$ satisfies the stochastic differential equation (3) with the given initial condition $|\psi_0\rangle$.

Under these assumptions it is a straightforward exercise in Ito calculus (see, for example, [4]) to show that $\langle\psi_t|\psi_t\rangle = 1$ for all $t \in [0, \infty)$. One is then led to the following basic result.

Proposition 1. *The Hamiltonian process $\{H_t\}$ is an $\{\mathcal{F}_t\}$-martingale, and the variance process $\{V_t\}$ is an $\{\mathcal{F}_t\}$-supermartingale.*

Proof. We need to show that $\{H_t\}$ satisfies

$$\mathbb{E}_s[H_t] = H_s, \tag{24}$$

and that $\{V_t\}$ satisfies

$$\mathbb{E}_s[V_t] \leqslant V_s, \tag{25}$$

where $\mathbb{E}_t[\cdot]$ denotes conditional expectation with respect to the σ-algebra $\mathcal{F}_t$. The validity of these properties can be established as follows. By an application of Ito's lemma to (4) and (5), we infer that

$$\mathrm{d}H_t = \sigma V_t \,\mathrm{d}W_t, \tag{26}$$

and that

$$\mathrm{d}V_t = -\sigma^2 V_t^2 \,\mathrm{d}t + \sigma \kappa_t \,\mathrm{d}W_t. \tag{27}$$

The process $\{\kappa_t\}$ defined here by

$$\kappa_t = \frac{\langle \psi_t | (\hat{H} - H_t)^3 | \psi_t \rangle}{\langle \psi_t | \psi_t \rangle} \tag{28}$$

measures the skewness of the energy distribution. Integrating (26) and (27) we deduce that

$$H_t = H_0 + \sigma \int_0^t V_u \,\mathrm{d}W_u, \tag{29}$$

and that

$$V_t = V_0 - \sigma^2 \int_0^t V_u^2 \,\mathrm{d}u + \sigma \int_0^t \kappa_u \,\mathrm{d}W_u. \tag{30}$$

Then on account of the relation

$$\mathbb{E}_s\left[\int_0^t b_u \,\mathrm{d}W_u\right] = \int_0^s b_u \,\mathrm{d}W_u \tag{31}$$

that holds for the stochastic integral of any $\{\mathcal{F}_t\}$-adapted process $\{b_t\}$ satisfying

$$\mathbb{E}\left[\int_0^t b_u^2 \,\mathrm{d}u\right] < \infty, \tag{32}$$

we deduce the martingale condition (24) from (29). This follows from the fact that $\{V_t\}$ is bounded. Similarly, it follows as a consequence of (30), and the fact that $\{\kappa_t\}$ is bounded, that

$$\mathbb{E}_s[V_t] = V_s - \sigma^2 \mathbb{E}_s\left[\int_s^t V_u^2 \,\mathrm{d}u\right], \tag{33}$$

which then implies the supermartingale condition (25). □

4. Convergence of the energy variance

In the case of the Schrödinger equation with a time-independent Hamiltonian, the energy process defined by (4) is constant. This is usually interpreted as the quantum mechanical expression of an energy conservation principle. The martingale relation (24) arising in the case of the energy-based stochastic Schrödinger equation can be viewed as a refinement of this principle.

The supermartingale property (25) satisfied by the variance process is the essence of what is meant by a *reduction* process. In the case of the Schrödinger equation with a time-independent Hamiltonian, the variance of the energy is a constant of the motion. In other words, not only is the expectation value of the energy fixed, so is the spread. On the other hand, the spread of the energy is reduced in the case of the stochastic dynamics of (3). In fact, the following result follows as a consequence of equation (30).

Proposition 2. *The asymptotic behaviour of the expectation of* $\{V_t\}$ *is given by*

$$\lim_{t\to\infty} \mathbb{E}[V_t] = 0. \tag{34}$$

Therefore, the variance process is a potential.

Proof. First we note that if X and Y are integrable random variables, and if $X \leqslant Y$ almost surely, then $\mathbb{E}[X] \leqslant \mathbb{E}[Y]$. It follows thus from the supermartingale condition (25) by use of the tower property that if $t \geqslant u$ then $\mathbb{E}[V_t] \leqslant \mathbb{E}[V_u]$. We note that

$$\mathbb{E}\left[\int_0^t b_u \, \mathrm{d}W_u\right] = 0 \tag{35}$$

for any $\{\mathcal{F}_t\}$-adapted process $\{b_t\}$ satisfying (32). Since the energy skewness process $\{\kappa_t\}$ is bounded, it therefore follows from (30) that

$$\begin{aligned}\mathbb{E}[V_t] &= V_0 - \sigma^2 \mathbb{E}\left[\int_0^t V_u^2 \, \mathrm{d}u\right] \\ &= V_0 - \sigma^2 \int_0^t \mathbb{E}\big[V_u^2\big] \, \mathrm{d}u.\end{aligned} \tag{36}$$

Here, we have used Fubini's theorem to interchange the expectation and the integration. As a consequence, we have the relation

$$\mathbb{E}[V_t] \leqslant V_0 - \sigma^2 \int_0^t (\mathbb{E}[V_u])^2 \, \mathrm{d}u, \tag{37}$$

since $\mathbb{E}\big[V_t^2\big] \geqslant (\mathbb{E}[V_t])^2$, which follows from Jensen's inequality. Now writing

$$v = \lim_{t\to\infty} \mathbb{E}[V_t], \tag{38}$$

let us suppose that $v \neq 0$. Because $\mathbb{E}[V_t]$ is a nonnegative, nonincreasing function of time, we have $v \leqslant \mathbb{E}[V_t]$. It follows from (37) that $\mathbb{E}[V_t] \leqslant V_0 - \sigma^2 v^2 t$, which, if $v \neq 0$, implies that $\mathbb{E}[V_t]$ vanishes at $t = V_0/\sigma^2 v^2$. However, this is incompatible with the assumption that $v \neq 0$; it follows that $v = 0$ and thus that $\{V_t\}$ is a potential. □

The same conclusion can be reached by a slightly different line of argument. Starting with (37), we use the fact that $\mathbb{E}[V_t] \leqslant \mathbb{E}[V_u]$ for $t \geqslant u$ to infer that $\mathbb{E}[V_t] \leqslant V_0 - \sigma^2 t (\mathbb{E}[V_t])^2$, which implies, on account of the positivity of $\mathbb{E}_t[V_t]$, that

$$\mathbb{E}[V_t] \leqslant \sqrt{\frac{V_0}{\sigma^2 t}}, \tag{39}$$

and hence the claim of the proposition.

Since V_t is nonnegative, proposition 2 implies that $\lim_{t\to\infty} V_t = 0$ almost surely, i.e. that reduction to a state of vanishing energy uncertainty occurs with probability 1.

5. Asymptotic properties of the energy

From the martingale convergence theorem for square-integrable martingales (see section 2), it follows that there exists a random variable H_∞ defined by

$$H_\infty = \lim_{t\to\infty} H_t, \tag{40}$$

which represents the terminal value of the energy once the reduction is complete. Thus, if we write

$$H_\infty = H_0 + \sigma \int_0^\infty V_u \, \mathrm{d}W_u, \tag{41}$$

it follows as a consequence of (31) that

$$H_t = \mathbb{E}_t[H_\infty]. \tag{42}$$

Thus, H_∞ has the property that it *closes* the martingale $\{H_t\}$. It then follows from (42) that the random variable H_t has the interpretation of being the $\mathcal{F}_t$-conditional expectation of the terminal value of the energy. In particular, we deduce that $H_0 = \mathbb{E}[H_\infty]$, which shows that the expectation value of the Hamiltonian in the initial state agrees with the expectation of the terminal value of the energy. This result can be viewed as a justification for the conventional interpretation of the expectation value of the Hamiltonian.

A similar result can be established in the case of the variance, which we now proceed to derive. In particular, writing (33) in the form

$$\mathbb{E}_t[V_T] = V_t - \sigma^2 \mathbb{E}_t\left[\int_t^T V_u^2\, \mathrm{d}u\right], \tag{43}$$

it follows that

$$\lim_{T\to\infty} \mathbb{E}_t[V_T] = V_t - \sigma^2 \lim_{T\to\infty} \mathbb{E}_t\left[\int_t^T V_u^2\, \mathrm{d}u\right]. \tag{44}$$

Since the variance of the energy is bounded, we can invoke the conditional form of the bounded convergence theorem to interchange the order of the limit and the expectation on the left-hand side of this equation. It follows from the fact that $\lim_{T\to\infty} V_T = 0$ almost surely that

$$V_t = \sigma^2 \lim_{T\to\infty} \mathbb{E}_t\left[\int_t^T V_u^2\, \mathrm{d}u\right]. \tag{45}$$

Now we interchange the order of the limit and the conditional expectation on the right-hand side of this equation by using the conditional form of the monotone convergence theorem and we deduce that

$$V_t = \sigma^2 \mathbb{E}_t\left[\int_t^\infty V_u^2\, \mathrm{d}u\right]. \tag{46}$$

On the other hand, it follows as a consequence of (29) and (41) that

$$H_\infty - H_t = \sigma \int_t^\infty V_u\, \mathrm{d}W_u, \tag{47}$$

and therefore, by use of the conditional Wiener–Ito isometry (22), that

$$\mathbb{E}_t[(H_\infty - H_t)^2] = \sigma^2 \mathbb{E}_t\left[\left(\int_t^\infty V_u\, \mathrm{d}W_u\right)^2\right] = \sigma^2 \mathbb{E}_t\left[\int_t^\infty V_u^2\, \mathrm{d}u\right]. \tag{48}$$

Equating the results (46) and (48) we obtain the fundamental relation:

Proposition 3. *Let $\{|\psi_t\rangle\}$ satisfy (3) and write H_∞ for the asymptotic value of the energy martingale $\{H_t\}$. Then the squared uncertainty of the energy in the state $|\psi_t\rangle$ is given by*

$$V_t = \mathbb{E}_t[(H_\infty - \mathbb{E}_t[H_\infty])^2]. \tag{49}$$

This relation shows that the random variable V_t has the interpretation of being the *conditional variance* of the *terminal* value of the energy. In particular, proposition 3 demonstrates that the initial squared energy uncertainty V_0 agrees with the variance of the terminal value of the energy. This fact can be viewed as a justification for the conventional interpretation of the energy uncertainty.

6. Asymptotic properties of observables that are compatible with the energy

We now proceed to derive a rather more general result that includes proposition 3 as a special case. Let us suppose $\hat{G}$ is any observable that commutes with $\hat{H}$ and write G_t and V_t^G for the mean and variance of $\hat{G}$ with respect to the random state $|\psi_t\rangle$. Thus $G_t = \langle\hat{G}\rangle_t$ and $V_t^G = \langle(\hat{G} - G_t)^2\rangle_t$, and by use of Ito's lemma we deduce as a consequence of (3) that

$$dG_t = \sigma\gamma_t\, dW_t, \tag{50}$$

and that

$$dV_t^G = -\sigma^2\gamma_t^2\, dt + \sigma\delta_t\, dW_t, \tag{51}$$

where

$$\gamma_t = \langle(\hat{G} - G_t)(\hat{H} - H_t)\rangle_t \tag{52}$$

and

$$\delta_t = \langle(\hat{G} - G_t)^2(\hat{H} - H_t)\rangle_t. \tag{53}$$

Now we shall show that $\{G_t\}$ is a martingale and investigate the nature of the conditional variance representation admitted by $\{V_t^G\}$. It follows from (50) and (51) that

$$G_t = G_0 + \sigma\int_0^t \gamma_u\, dW_u \tag{54}$$

and that

$$V_t^G = V_0^G - \sigma^2\int_0^t \gamma_u^2\, du + \sigma\int_0^t \delta_u\, dW_u. \tag{55}$$

Thus, since $\{\gamma_t\}$ and $\{\delta_t\}$ are bounded, we see that $\{G_t\}$ is an $\{\mathcal{F}_t\}$-martingale and that $\{V_t^G\}$ is an $\{\mathcal{F}_t\}$-supermartingale. Therefore, by the martingale convergence theorem (see section 2) there exist random variables G_∞ and V_∞^G such that

$$G_\infty = G_0 + \sigma\int_0^\infty \gamma_u\, dW_u \tag{56}$$

and

$$V_\infty^G = V_0^G - \sigma^2\int_0^\infty \gamma_u^2\, du + \sigma\int_0^\infty \delta_u\, dW_u. \tag{57}$$

Taking the conditional expectation of each side of this equation with respect to $\mathcal{F}_t$, we deduce that

$$\mathbb{E}_t\big[V_\infty^G\big] = V_0^G - \sigma^2\mathbb{E}_t\left[\int_0^\infty \gamma_u^2\, du\right] + \sigma\int_0^t \delta_u\, dW_u. \tag{58}$$

Solving (58) for V_0^G and substituting the result into (55) we see that

$$\begin{aligned} V_t^G &= \mathbb{E}_t\big[V_\infty^G\big] + \sigma^2\mathbb{E}_t\left[\int_t^\infty \gamma_u^2\, du\right] \\ &= \mathbb{E}_t\big[V_\infty^G\big] + \sigma^2\mathbb{E}_t\left[\left(\int_t^\infty \gamma_u\, dW_u\right)^2\right], \end{aligned} \tag{59}$$

by use of the conditional Wiener–Ito isometry. Making use of the fact that

$$G_\infty = G_t + \sigma\int_t^\infty \gamma_u\, dW_u, \tag{60}$$

which follows from (54) and (56), we obtain the following fundamental relations governing the dynamics of $\{G_t\}$ and $\{V_t^G\}$:

$$G_t = \mathbb{E}_t[G_\infty] \tag{61}$$

and

$$V_t^G = \mathbb{E}_t\left[V_\infty^G\right] + \mathbb{E}_t[(G_\infty - G_t)^2]. \tag{62}$$

Equation (61) shows that the martingale $\{G_t\}$ *closes*, and hence the relation $G_0 = \mathbb{E}[G_\infty]$ allows us to identify the initial expectation value G_0 with the expectation of the result obtained for the random variable G_∞. Equation (62) then has a natural interpretation as a *conditional variance relation*. If the Hamiltonian has a nondegenerate spectrum, then the terminal state is necessarily both an eigenstate of $\hat{H}$ and $\hat{G}$, and V_∞^G vanishes. On the other hand, if $\hat{H}$ has a degenerate spectrum, then the terminal value of $\hat{H}$ will not necessarily be an eigenvalue of $\hat{G}$. In that case, the random variable V_∞^G is nonvanishing and takes the value

$$V_\infty^G = \langle\phi_i|\hat{G}^2|\phi_i\rangle - \langle\phi_i|\hat{G}|\phi_i\rangle^2 \tag{63}$$

with probability

$$\pi_i = |\langle\psi_0|\phi_i\rangle|^2, \tag{64}$$

where $|\phi_i\rangle$ is the normalized Lüders state[3] corresponding to the eigenvalue E_i of $\hat{H}$, given the initial state $|\psi_0\rangle$, as defined in section 1. We recall that when $\hat{H}$ has a degenerate spectrum, the collapse of the wavefunction induced by (3) necessarily leads to one of the Lüders states, as shown, e.g., in [4].

It is interesting to observe that, while the variance process $\{V_t\}$ associated with the Hamiltonian is a potential, the variance $\{V_t^G\}$ for $\hat{G}$, although a supermartingale, is not necessarily a potential unless $\hat{H}$ has a nondegenerate spectrum. Physically this is because a reduction of the energy induces a complete reduction of a compatible observable only if the energy spectrum is nondegenerate.

7. On the generality of the dynamical equation

Before embarking on an account of our approach to the solution of the stochastic differential equation (3), it will be useful to set this stochastic equation in the context of a more general family of possible dynamical laws for the state-vector process $\{|\psi_t\rangle\}$. The idea then will be to see what specific additional physical assumptions are needed to imply that the dynamics should take the form (3).

We shall assume as before that $\{|\psi_t\rangle\}_{0\leqslant t<\infty}$ is a continuous stochastic process defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$ taking values in a finite-dimensional complex Hilbert space. For the dynamics of $\{|\psi_t\rangle\}$, we write

$$\mathrm{d}|\psi_t\rangle = \hat{\mu}_t|\psi_t\rangle\,\mathrm{d}t + \hat{\sigma}_t|\psi_t\rangle\,\mathrm{d}W_t, \tag{65}$$

[3] See [4, 19, 22]. We remark, incidentally, that the Lüders state has the following geometrical interpretation. In the case of a Hilbert space of dimension $n+1$, the corresponding space of pure states is the complex projective space $\mathbb{C}\mathrm{P}^n$. If the Hilbert subspace of state vectors of some given energy E_i has dimension $k+1$, then the corresponding space of pure states of that energy is a projective hyperplane D^k of dimension k. The complex conjugate of D^k is a hyperplane $\bar{D}^{n-k-1}$ of dimension $n-k-1$. Clearly D^k and $\bar{D}^{n-k-1}$ do not intersect. The initial state vector $|\psi_0\rangle$ corresponds to a point $\psi_0 \in \mathbb{C}\mathrm{P}^n$. Therefore, the join of ψ_0 and D^k is a hyperplane Q^{k+1} of dimension $k+1$ which intersects the hyperplane $\bar{D}^{n-k-1}$ at a single point $\bar{\psi}_i$. Now take the join of ψ_0 and $\bar{\psi}_i$. The resulting line clearly lies in hyperplane Q^{k+1} and thus hits the hyperplane D^k at a single point, and this point is the Lüders state ϕ_i. The interpretation of $\bar{\psi}_i$, on the other hand, is as follows: if a measurement is made to determine simply whether the energy is E_i or not, then in the event of a negative result the new state of the system will be the point $\bar{\psi}_i$.

where $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion. We assume that $\{|\psi_t\rangle\}$ is adapted to $\{\mathcal{F}_t\}$ and that so are the operator-valued processes $\{\hat{\mu}_t\}$ and $\{\hat{\sigma}_t\}$. We call $\{\hat{\mu}_t\}$ and $\{\hat{\sigma}_t\}$ the operator-valued drift and volatility of $\{|\psi_t\rangle\}$.

Our first requirement will be that $\{\hat{\mu}_t\}$ and $\{\hat{\sigma}_t\}$ must be chosen such that the normalization of $|\psi_t\rangle$ is preserved for all t. We note that the conjugate of (65) is

$$\mathrm{d}\langle\psi_t| = \langle\psi_t|\hat{\mu}_t^\dagger\,\mathrm{d}t + \langle\psi_t|\hat{\sigma}_t^\dagger\,\mathrm{d}W_t. \tag{66}$$

By virtue of the Ito product rule we have

$$\mathrm{d}\langle\psi_t|\psi_t\rangle = (\mathrm{d}\langle\psi_t|)|\psi_t\rangle + \langle\psi_t|(\mathrm{d}|\psi_t\rangle) + (\mathrm{d}\langle\psi_t|)(\mathrm{d}|\psi_t\rangle), \tag{67}$$

and thus by use of (65) and (66) we obtain

$$\frac{\mathrm{d}\langle\psi_t|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} = \left(\left\langle\hat{\mu}_t^\dagger + \hat{\mu}_t\right\rangle_t + \left\langle\hat{\sigma}_t^\dagger\hat{\sigma}_t\right\rangle_t\right)\mathrm{d}t + \left\langle\hat{\sigma}_t^\dagger + \hat{\sigma}_t\right\rangle_t\mathrm{d}W_t, \tag{68}$$

where for brevity we use the convenient notation

$$\langle\hat{X}_t\rangle_t = \frac{\langle\psi_t|\hat{X}_t|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \tag{69}$$

for the expectation value at time t of any operator process $\{\hat{X}_t\}$. Therefore, the normalization condition for $|\psi_t\rangle$ is ensured if the operators $\hat{\mu}_t^\dagger + \hat{\mu}_t + \hat{\sigma}_t^\dagger\hat{\sigma}_t$ and $\hat{\sigma}_t^\dagger + \hat{\sigma}_t$ have vanishing expectation values with respect to $|\psi_t\rangle$. It is a straightforward exercise to verify that the most general expressions for the drift $\hat{\mu}$ and the volatility $\hat{\sigma}$ satisfying these conditions are

$$\hat{\mu}_t = -\mathrm{i}\hat{H}_t - \tfrac{1}{2}\hat{\sigma}_t^\dagger\hat{\sigma}_t + \hat{J}_t - \langle\hat{J}_t\rangle_t, \tag{70}$$

and

$$\hat{\sigma}_t = \mathrm{i}\hat{K}_t + \hat{L}_t - \langle\hat{L}_t\rangle_t, \tag{71}$$

where $\{\hat{H}_t\}$, $\{\hat{J}_t\}$, $\{\hat{K}_t\}$ and $\{\hat{L}_t\}$ are arbitrary Hermitian operator-valued processes.

It should be evident that the process (3) is obtained if (a) we let $\hat{K}_t$ and $\hat{J}_t$ vanish for all t, (b) we let $\hat{L}_t = \frac{1}{2}\sigma\hat{H}_t$ for all t, where σ is a parameter, and (c) we assume that $\{\hat{H}_t\}$ is time independent. Let us investigate therefore the nature of the additional physical conditions that we need to impose on the general norm-preserving dynamics in order to ensure that the energy-based model is obtained in accordance with these specifications.

The general norm-preserving model contains the four operator-valued processes $\{\hat{H}_t\}$, $\{\hat{J}_t\}$, $\{\hat{K}_t\}$ and $\{\hat{L}_t\}$. One can think of these operators as representing properties of the physical environment in which the quantum system exists. In general, the environment is changing in a random time-dependent manner. We shall make the simplifying assumption of a 'stationary' environment so that $\{\hat{H}_t\}$, $\{\hat{J}_t\}$, $\{\hat{K}_t\}$ and $\{\hat{L}_t\}$ are now replaced by time-independent operators $\hat{H}$, $\hat{J}$, $\hat{K}$ and $\hat{L}$. Thus, our first assumption is that the environment is in a state of stationary equilibrium.

We give the operator $\hat{H}$ the usual interpretation as representing the total energy of the system. This is justified by the fact that if $\hat{K}$, $\hat{J}$ and $\hat{L}$ are set to zero then the conventional Schrödinger equation is recovered. Next we make an assumption that might be called the 'universality of the Hamiltonian'. This is based on the observation that the Hamiltonian is the only observable that *must* exist as an element of the dynamics of a quantum system. If our dynamical law is to be universally applicable to any quantum system, then the only observable that can enter the discussion is $\hat{H}$, and thus we must require that $\hat{K}$, $\hat{J}$ and $\hat{L}$ are all functions of the Hamiltonian. Thus, in effect, we are asking that the system should act as its own environment. It is with this assumption that an element of nonlinearity enters the dynamics.

Our final physical requirement is that energy should be conserved in some suitable sense. Now in ordinary quantum mechanics with a time-independent Hamiltonian, the expectation value of the Hamiltonian is a constant of the motion. This relation is usually interpreted by physicists with a certain looseness of language to mean 'conservation of energy', but what it means really is conservation of the expectation value of the energy. In a situation where the state is undergoing random changes, the expectation value of the energy will also change randomly. We can, nonetheless, impose a slightly weaker condition of conservation appropriate to this situation by requiring that the process $\{H_t\}_{0\leqslant t<\infty}$ should satisfy the martingale relation

$$\mathbb{E}[H_t|\mathcal{F}_s] = H_s \tag{72}$$

for $s \leqslant t$. This relation states that the conditional expectation of the expectation value of the energy at time t, with respect to the σ-algebra $\mathcal{F}_s$, is the expectation value of the energy at time s. This is the sense in which the martingale relation provides a characterization of the principle of energy conservation.

It follows therefore that we need to analyse the process $\{H_t\}$, defined as in (4), and require that its drift should vanish. This will ensure that $\{H_t\}$ is a martingale and that energy is conserved. In fact, we shall impose a somewhat stronger condition. Let $f(x)$ denote a bounded function and write $\hat{f} = f(\hat{H})$. We shall require that for any such operator $\hat{f}$ the corresponding expectation-value process $\{f_t\}$ defined by

$$f_t = \frac{\langle\psi_t|f(\hat{H})|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \tag{73}$$

should be an $\{\mathcal{F}_t\}$-martingale. This corresponds to the requirement that not only is the energy conserved in the sense discussed above but so is the observable associated with any function of the energy. With this condition in place we have a suitably general and robust representation of the principle of energy conservation. If we take the stochastic differential of f_t in (73), then by use of the Ito calculus we find that

$$\mathrm{d}f_t = 2(\langle\hat{J}\hat{f}\rangle_t - \langle\hat{J}\rangle_t\langle\hat{f}\rangle_t)\,\mathrm{d}t + 2(\langle\hat{L}\hat{f}\rangle_t - \langle\hat{L}\rangle_t\langle\hat{f}\rangle_t)\,\mathrm{d}W_t. \tag{74}$$

Now the martingale condition on $\{f_t\}$ implies that the drift of $\{f_t\}$ in (74) must vanish. In other words, we require that the covariance of the two operators $\hat{J}$ and $\hat{f}$ should vanish for any choice of the function $f(x)$. Thus in particular if we set $\hat{f} = \hat{J}$ then it follows that the uncertainty of $\hat{J}$ must vanish in the state $|\psi_t\rangle$, and hence without loss of generality we may assume that $\hat{J}$ is a constant multiple of the identity matrix, and therefore drops out of the dynamics.

Finally, we consider the roles of $\hat{K}$ and $\hat{L}$ in the expression for $\hat{\sigma}_t$ in (55). To this end, we shall examine the dynamics of the squared uncertainty of the operator $\hat{H}$ in the state $|\psi_t\rangle$. Now so far we have through our physical considerations specialized the general dynamics (65) to the particular case

$$\mathrm{d}|\psi_t\rangle = \left(-\mathrm{i}\hat{H} - \tfrac{1}{2}\hat{\sigma}_t^\dagger\hat{\sigma}_t\right)|\psi_t\rangle\,\mathrm{d}t + \hat{\sigma}_t|\psi_t\rangle\,\mathrm{d}W_t, \tag{75}$$

where

$$\hat{\sigma}_t = \mathrm{i}\hat{K} + \hat{L} - \langle\hat{L}\rangle_t, \tag{76}$$

and $\hat{H}$, $\hat{K}$ and $\hat{L}$ are time independent and Hermitian, with the further provision that $\hat{K}$ and $\hat{L}$ are both given by functions of $\hat{H}$. We shall call (75) the general stationary energy-based stochastic Schrödinger equation.

Let us therefore investigate the extent to which the general stationary energy-based dynamics (75) necessarily leads to state reduction. Writing

$$V_t = \langle\hat{H}^2\rangle_t - \langle\hat{H}\rangle_t^2 \tag{77}$$

for the variance of $\hat{H}$ with respect to the state $|\psi_t\rangle$, we obtain

$$\mathrm{d}V_t = \mathrm{d}\langle\hat{H}^2\rangle_t - 2\langle\hat{H}\rangle_t\,\mathrm{d}\langle\hat{H}\rangle_t - (\mathrm{d}\langle\hat{H}\rangle_t)^2. \tag{78}$$

Now making use of the fact that both $\{\langle \hat{H}^2\rangle_t\}$ and $\{\langle \hat{H}\rangle_t\}$ are martingales (cf [5]), we immediately infer that $\{V_t\}$ is a supermartingale. In fact, a calculation gives

$$\mathrm{d}V_t = -4\langle(\hat{H}-\langle H\rangle_t)(\hat{L}-\langle L\rangle_t)\rangle_t^2\,\mathrm{d}t + 2\langle(\hat{H}-\langle H\rangle_t)^2(\hat{L}-\langle L\rangle_t)\rangle_t\,\mathrm{d}W_t. \tag{79}$$

It is apparent from (79) that the drift of the energy variance process is negative in the general stochastic extension of the Schrödinger equation given by (75). This demonstrates that *the presence of some element of state reduction or relaxation is a generic feature of the dynamics of (75), regardless of the specific choice of the functions determining $\hat{K}$ and $\hat{L}$.*

This result offers some support to the proposal put forward in [3] that dynamic reduction in quantum theory might be an 'emergent' phenomenon.

In fact, a short calculation establishes that under the general energy-based dynamics (75), the variance process $\{V_t^L\}$ associated with the operator $\hat{L}$, defined by $V_t^L = \langle(\hat{L}-\langle\hat{L}\rangle_t)^2\rangle_t$, satisfies the conditions of being a potential, and admits the following representation as a conditional variance:

$$V_t^L = \mathbb{E}_t[(L_\infty - L_t)^2], \tag{80}$$

where L_∞ denotes the terminal limiting value of the martingale $\{L_t\}$ defined by $L_t = \langle\hat{L}\rangle_t$.

Thus, provided the eigenstates of $\hat{L}$ are also eigenstates of $\hat{H}$, then (75) necessarily implies a reduction to energy eigenstates. In what follows, we shall therefore make the simplest choice that ensures this condition, namely, $\hat{K}=0$ and $\hat{L}=\frac{1}{2}\sigma\hat{H}$, where σ is a parameter. Nevertheless, we see that in a general setting there is scope for some variation in the dynamics of the state vector from that appearing in (3). In particular, we can also consider dropping the stationarity condition. Later in this paper we present a useful example of a nonstationary dynamical law.

8. Conditional probabilities for reduction

An important special case of the situation described in section 6 arises when the observable $\hat{G}$ corresponds to the projection operator $\hat{\Pi}_i$ onto the subspace of states with energy E_i (cf [4, 2]). In this case, we have the relations $\hat{H}\hat{\Pi}_i = \hat{\Pi}_i\hat{H}$, $\hat{\Pi}_i\hat{\Pi}_j = \delta_{ij}\hat{\Pi}_i$, $\sum_i \hat{\Pi}_i = 1$ and $\sum_i E_i\hat{\Pi}_i = \hat{H}$. The spectrum of $\hat{H}$ may or may not be degenerate.

Writing $\pi_{it} = \langle\hat{\Pi}_i\rangle_t$ for the expectation value of the operator $\hat{\Pi}_i$ in the state $|\psi_t\rangle$, we deduce as a consequence of the results of section 6 that

$$\mathrm{d}\pi_{it} = \sigma\pi_{it}(E_i - H_t)\,\mathrm{d}W_t, \tag{81}$$

and that

$$\mathrm{d}v_{it} = -\sigma^2\pi_{it}^2(E_i-H_t)^2\,\mathrm{d}t + \sigma\pi_{it}(1-2\pi_{it})(E_i-H_t)\,\mathrm{d}W_t. \tag{82}$$

Here v_{it} denotes the variance of the operator $\hat{\Pi}_i$ in the state $|\psi_t\rangle$. We note that in the case of a projection operator the variance takes the simple form

$$v_{it} = \pi_{it}(1-\pi_{it}). \tag{83}$$

The random variable π_{it} has the interpretation of being the conditional probability that reduction to a state with energy E_i will occur. In particular, the initial quantities $\pi_i = \pi_{i0}$ are the Dirac transition probabilities from the initial state $|\psi_0\rangle$ to a state with energy E_i.

It is evident that the process $\{\pi_{it}\}$ is a martingale and that this martingale is closed by the random variable

$$\pi_{i\infty} = \mathbf{1}_{\{H_\infty = E_i\}}. \tag{84}$$

That is to say,

$$\pi_{it} = \mathbb{E}_t[\pi_{i\infty}]. \tag{85}$$

As a consequence we see that v_{it} can be written in the form

$$v_{it} = \mathbb{E}_t[(\pi_{i\infty} - \mathbb{E}_t[\pi_{i\infty}])^2]. \tag{86}$$

In other words, v_{it} can be interpreted as the conditional variance of the indicator function for collapse to a state of energy E_i. Equation (86) follows immediately from (83) if we make use of the fact that the terminal indicator function for the energy E_i satisfies $(\pi_{i\infty})^2 = \pi_{i\infty}$. We see therefore that $\{v_{it}\}$ is a potential.

Now we proceed to derive another expression for $\{\pi_{it}\}$ that will play a key role in the developments that follow. It is well known from the theory of stochastic differential equations that an equation of the form (81) can be integrated. If a positive process $\{X_t\}$ satisfies an equation of the form

$$\mathrm{d}X_t = \alpha_t X_t \,\mathrm{d}W_t, \tag{87}$$

and if $\int_0^t \alpha_s^2 \,\mathrm{d}s < \infty$ almost surely for all $t \in [0, \infty)$, then we can write

$$X_t = X_0 \exp\left(\int_0^t \alpha_u \,\mathrm{d}W_u - \frac{1}{2}\int_0^t \alpha_u^2 \,\mathrm{d}u\right), \tag{88}$$

where X_0 is the initial value of the process. If $\{\alpha_t\}$ itself depends in some way on $\{X_t\}$ then one cannot say that (88) 'solves' (87). In that situation (88) should be regarded as an integral representation of the stochastic differential equation (87). Nevertheless, we may be able to extract useful information about the process $\{X_t\}$ by expressing it in this form. In the present case we can integrate (81) to obtain

$$\pi_{it} = \pi_i \exp\left(\sigma \int_0^t (E_i - H_u)\,\mathrm{d}W_u - \frac{1}{2}\sigma^2 \int_0^t (E_i - H_u)^2 \,\mathrm{d}u\right). \tag{89}$$

After some straightforward algebraic rearrangement this can be put in the form

$$\pi_{it} = \frac{\pi_i \exp\left[\sigma E_i\left(W_t + \sigma \int_0^t H_u \,\mathrm{d}u\right) - \frac{1}{2}\sigma^2 E_i^2 t\right]}{\exp\left(\sigma \int_0^t H_u \,\mathrm{d}W_u + \frac{1}{2}\sigma^2 \int_0^t H_u^2 \,\mathrm{d}u\right)}. \tag{90}$$

A further simplification is then achieved if we introduce the $\{\mathcal{F}_t\}$-adapted process $\{\xi_t\}_{0\leqslant t<\infty}$ defined by the relation

$$\xi_t = \sigma \int_0^t H_u \,\mathrm{d}u + W_t. \tag{91}$$

The process $\{\xi_t\}$ is evidently a Brownian motion with drift. Making use of the relation

$$\mathrm{d}\xi_t = \sigma H_t \,\mathrm{d}t + \mathrm{d}W_t, \tag{92}$$

we can then put (90) in the form

$$\pi_{it} = \frac{\pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\exp\left(\sigma \int_0^t H_s \,\mathrm{d}\xi_s - \frac{1}{2}\sigma^2 \int_0^t H_s^2 \,\mathrm{d}s\right)}. \tag{93}$$

Finally, we note that since $\sum_i \pi_{it} = 1$, equation (93) leads us to the following identity:

$$\exp\left(\sigma \int_0^t H_u \,\mathrm{d}\xi_u - \frac{1}{2}\int_0^t H_u^2 \,\mathrm{d}u\right) = \sum_i \pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right). \tag{94}$$

Inserting this relation into (93) we obtain the following result.

Proposition 4. *The conditional probability process $\{\pi_{it}\}$ for reduction to a state of energy E_i takes the form*

$$\pi_{it} = \frac{\pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\sum_i \pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}, \tag{95}$$

where $\xi_t = W_t + \sigma \int_0^t H_u \, \mathrm{d}u$. *The energy expectation process* $\{H_t\}$ *is given by*

$$H_t = \frac{\sum_i \pi_i E_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\sum_i \pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}, \tag{96}$$

and the energy variance process $\{V_t\}$ *is given by*

$$V_t = \frac{\sum_i \pi_i (E_i - H_t)^2 \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\sum_i \pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}. \tag{97}$$

We observe, incidentally, that it suffices to specify the value of ξ_t to determine π_{it}, H_t and V_t. In other words, the random behaviour of these quantities is specified entirely through their dependence on ξ_t.

9. Information theoretic interpretation of the reduction process

Given the conditional probability π_{it} for reduction to an energy eigenstate with energy E_i, we can consider the associated information entropy S_t. Since the conditional probability approaches the indicator function (84) asymptotically, we expect the associated entropy to decrease on average. This idea can be put into more precise terms as follows:

Proposition 5. *The Shannon entropy process* $\{S_t\}$ *associated with the conditional probability process* $\{\pi_{it}\}$ *is a potential.*

The Shannon entropy (or information entropy) associated with π_{it} is defined by the expression

$$S_t = -\sum_i \pi_{it} \ln \pi_{it}. \tag{98}$$

This entropy is associated in a natural way with the *random density matrix* process defined by the conditional expectation of the terminal state of the system:

$$\hat{R}_t = \mathbb{E}_t[|\psi_\infty\rangle\langle\psi_\infty|]. \tag{99}$$

For each value of t, clearly $\hat{R}_t$ is positive semi-definite and has unit trace. It should also be evident that

$$S_t = -\,\mathrm{tr}(\hat{R}_t \ln \hat{R}_t). \tag{100}$$

We note that the process $\{\hat{R}_t\}$ is distinct from the process $\{\hat{\rho}_t\}$ defined by

$$\hat{\rho}_t = \mathbb{E}[|\psi_t\rangle\langle\psi_t|], \tag{101}$$

which is deterministic in t. Thus, the state $\hat{R}_t$ represents the best conditional estimate of the terminal state of the system, whereas the state $\hat{\rho}_t$ represents the initial unconditional expectation of the state that the system will be in at time t. Evidently we have $\hat{R}_0 = \hat{\rho}_\infty$. For clarity let us call $\hat{\rho}_t$ the *von Neumann state* and $\hat{R}_t$ the *Shannon state*.

Now if the initial state of the system is a pure state $|\psi_0\rangle$, with minimum von Neumann entropy, then as the reduction proceeds the von Neumann state evolves into a mixed state $\hat{\rho}_t$ with higher entropy. Therefore, the von Neumann entropy $-\mathrm{tr}(\hat{\rho}_t \ln \hat{\rho}_t)$ associated with the mixed state $\hat{\rho}_t$ increases from zero to the terminal value $-\mathrm{tr}(\hat{\rho}_\infty \ln \hat{\rho}_\infty) = -\sum_i \pi_i \ln \pi_i$. On the other hand, the entropy of the initial Shannon state $\hat{R}_0$ is $-\sum_i \pi_i \ln \pi_i$ and the entropy of the terminal Shannon state $\hat{R}_\infty$ is zero.

Thus, the evolution $\{\hat{\rho}_t\}$ of the von Neumann state describes the increase in ignorance that results in the statistical description of the system as time moves forward; whereas the evolution

$\{\hat{R}_t\}$ of the Shannon state describes the increase in information that results as the measurement outcome is revealed. To put the matter another way, the entropy S_t associated with the Shannon state $\hat{R}_t$ is the negative of the information content generated by the information flow $\{\mathcal{F}_s\}_{0\leqslant s\leqslant t}$ up to time t. In particular, we expect the Shannon entropy S_t to decrease on average, because more information is gained as the collapse process progresses. Finally, when the state has reached an eigenstate, $\hat{R}_\infty$ becomes a pure state. The proposition above, which we now proceed to prove, asserts that this is indeed the case.

Proof of proposition 5. To begin, we determine the dynamical equation satisfied by the entropy process $\{S_t\}$. We note that, as a consequence of the dynamical equation (81) for the conditional probability process, and the use of Ito's lemma, we have

$$\mathrm{d}(\ln \pi_{it}) = -\tfrac{1}{2}\sigma^2(E_i - H_t)^2\,\mathrm{d}t + \sigma(E_i - H_t)\,\mathrm{d}W_t. \tag{102}$$

It follows, by another application of Ito's lemma, that

$$\mathrm{d}S_t = -\frac{1}{2}\sigma^2 V_t\,\mathrm{d}t - \sigma\left(\sum_i E_i\pi_{it}\ln\pi_{it} - H_t S_t\right)\mathrm{d}W_t, \tag{103}$$

where $\{H_t\}$ is the energy process and $\{V_t\}$ is the energy variance process. We observe that the volatility of $\{S_t\}$ is the covariance of the energy and the logarithm of the conditional probability. Since the drift of $\{S_t\}$ is strictly negative we see that the entropy process is a supermartingale.

To show that $\{S_t\}$ is a potential we need to show that $\lim_{t\to\infty}\mathbb{E}[S_t]$ vanishes. Because the conditional probabilities $\{\pi_{it}\}$ are bounded in the range $0 \leqslant \pi_{it} \leqslant 1$, the entropy is positive, and is also bounded, and thus $\lim_{t\to\infty}\mathbb{E}[S_t] = \mathbb{E}[\lim_{t\to\infty} S_t]$ by virtue of the bounded convergence theorem. On the other hand, (84) implies that $\pi_{i\infty}$ is unity if the terminal energy is E_i and zero otherwise. Therefore, $\lim_{t\to\infty} S_t = 0$ almost surely, and that establishes the result. □

The fact that $\{S_t\}$ is a potential leads to the following observation concerning the Shannon entropy and energy fluctuations during the reduction process.

Proposition 6. *The Shannon entropy process $\{S_t\}$ is given by the conditional expectation of the integrated future energy fluctuation level:*

$$S_t = \frac{1}{2}\sigma^2\mathbb{E}_t\left[\int_t^\infty V_s\,\mathrm{d}s\right]. \tag{104}$$

Proof. To derive this result we integrate the dynamical equation (103) satisfied by the entropy to deduce that

$$S_T = S_0 - \frac{1}{2}\sigma^2\int_0^T V_s\,\mathrm{d}s - \sigma\int_0^T\left(\sum_i E_i\pi_{is}\ln\pi_{is} - H_s S_s\right)\mathrm{d}W_s. \tag{105}$$

Taking the conditional expectation of this relation we infer, after some rearrangement of terms, that

$$\mathbb{E}_t[S_T] = S_t - \frac{1}{2}\sigma^2\mathbb{E}_t\left[\int_t^T V_s\,\mathrm{d}s\right]. \tag{106}$$

The identity (104) then follows from the fact that $\{S_t\}$ is a potential. □

It is interesting to note, incidentally, that if we let $t \to 0$ in (106), we obtain the following formula for the cumulative energy fluctuation during the collapse process:

$$\frac{1}{2}\sigma^2\int_0^\infty \mathbb{E}[V_s]\,\mathrm{d}s = S_0. \tag{107}$$

If the information entropy associated with the initial transition probabilities $\{\pi_i\}$ is large, so that the initial pure state $|\psi_0\rangle$ is a highly homogenized superposition of the energy eigenstates, one would expect the energy fluctuations during the reduction process to be large. Conversely, if this entropy is small, so that the initial state is close to one or a few of the eigenstates, then the energy fluctuations during the reduction process should be small. Proposition 6 makes this intuition precise. In particular, the right-hand side of (107) measures the entropic uncertainty of the initial energy dispersion (see, e.g., [14]) and is independent of the energy spectrum of the system.

10. Remarks on the ancillary linear dynamics for the state vector

One of the main goals of this paper is to present in detail a general method for obtaining the solution to the dynamical equation (3). Before embarking upon this, however, we shall first consider the properties of the linear stochastic differential equation

$$\mathrm{d}|\phi_t\rangle = -\mathrm{i}\hat{H}|\phi_t\rangle\,\mathrm{d}t - \tfrac{1}{8}\sigma^2\hat{H}^2|\phi_t\rangle\,\mathrm{d}t + \tfrac{1}{2}\sigma\hat{H}|\phi_t\rangle\,\mathrm{d}\xi_t, \tag{108}$$

and study the relation of this equation to (3). The stochastic differential equation (108), which we shall call the ancillary equation, plays an important role in the analysis of (3). In this section we shall also introduce some change-of-measure formulae that will be applied in later sections of the paper.

In the analysis that follows in this section, the process $\{|\phi_t\rangle\}_{0\leqslant t<\infty}$ is to be understood as defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0\leqslant t<\infty}$ with respect to which $\{\xi_t\}_{0\leqslant t<\infty}$ is a standard Brownian motion. The precise relation of the measure $\mathbb{Q}$ appearing here to the measure $\mathbb{P}$ introduced earlier will be specified shortly, as will the relation between the processes $\{\xi_t\}$ and $\{W_t\}$. In particular, the process $\{\xi_t\}$ introduced in this section has no *a priori* relation to the process $\{\xi_t\}$ with the same name introduced in section 8, though in what follows the connection between these processes is made precise.

The solution to the ancillary equation (108) is given by

$$|\phi_t\rangle = \exp\left(-\mathrm{i}\hat{H}t + \tfrac{1}{2}\sigma\hat{H}\xi_t - \tfrac{1}{4}\sigma^2\hat{H}^2 t\right)|\phi_0\rangle, \tag{109}$$

where $|\phi_0\rangle$ is a prescribed initial state, normalized to unity. The fact that (109) implies (108) can be verified by a direct application of Ito's lemma

$$\mathrm{d}|\phi_t\rangle = |\dot{\phi}(\xi_t, t)\rangle\,\mathrm{d}t + |\phi'(\xi_t, t)\rangle\,\mathrm{d}\xi_t + \tfrac{1}{2}|\phi''(\xi_t, t)\rangle(\mathrm{d}\xi_t)^2, \tag{110}$$

with $|\phi_t\rangle = |\phi(\xi_t, t)\rangle$, where the function $|\phi(\xi, t)\rangle$ is defined by

$$|\phi(\xi, t)\rangle = \exp\left(-\mathrm{i}\hat{H}t + \tfrac{1}{2}\sigma\hat{H}\xi - \tfrac{1}{4}\sigma^2\hat{H}^2 t\right)|\phi_0\rangle. \tag{111}$$

The dot and prime in (110) denote differentiation with respect to t and ξ, respectively.

We see as a consequence of (109) that the squared norm of $|\phi_t\rangle$ takes the form

$$\langle\phi_t|\phi_t\rangle = \langle\phi_0|\exp\left(\sigma\hat{H}\xi_t - \tfrac{1}{2}\sigma^2\hat{H}^2 t\right)|\phi_0\rangle. \tag{112}$$

Now writing $\hat{H} = \sum_i E_i\hat{\Pi}_i$, where $\hat{\Pi}_i$ as before denotes the projection operator onto the Hilbert subspace for which $\hat{H}$ takes the value E_i, we have

$$\langle\phi_t|\phi_t\rangle = \sum_i \pi_i \exp\left(\sigma E_i\xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right). \tag{113}$$

In other words, $\langle\phi_t|\phi_t\rangle$ can be expressed as a *weighted sum of geometric Brownian motions*. Here, π_i as before signifies the Dirac transition probability

$$\pi_i = \frac{\langle\phi_0|\hat{\Pi}_i|\phi_0\rangle}{\langle\phi_0|\phi_0\rangle} \tag{114}$$

from the initial state $|\phi_0\rangle$ to the Lüders state $|\phi_i\rangle$ associated with the initial state $|\phi_0\rangle$ and the eigenvalue E_i. It follows immediately by virtue of the properties of geometric Brownian motion that the process $\{\langle\phi_t|\phi_t\rangle\}$ is a martingale in the $\mathbb{Q}$-measure, satisfying

$$\mathbb{E}_s^{\mathbb{Q}}[\langle\phi_t|\phi_t\rangle] = \langle\phi_s|\phi_s\rangle. \tag{115}$$

This result can also be seen to follow directly from (112), by an application of Ito's lemma, which shows the squared norm of $|\phi_t\rangle$ satisfies the dynamical equation

$$\mathrm{d}\langle\phi_t|\phi_t\rangle = \sigma H_t \langle\phi_t|\phi_t\rangle\, \mathrm{d}\xi_t, \tag{116}$$

where the process $\{H_t\}$ is defined by

$$H_t = \frac{\langle\phi_0|\hat{H}\exp\left(\sigma\hat{H}\xi_t - \frac{1}{2}\sigma^2\hat{H}^2 t\right)|\phi_0\rangle}{\langle\phi_0|\exp\left(\sigma\hat{H}\xi_t - \frac{1}{2}\sigma^2\hat{H}^2 t\right)|\phi_0\rangle}. \tag{117}$$

The stochastic differential equation (116) can then be put in an integral form to give a useful alternative expression for the squared norm:

$$\langle\phi_t|\phi_t\rangle = \exp\left(\sigma\int_0^t H_u\,\mathrm{d}\xi_u - \frac{1}{2}\sigma^2\int_0^t H_u^2\,\mathrm{d}u\right). \tag{118}$$

Our intention is to show that the process $\{H_t\}$ defined in (117) can in fact be identified with the energy process defined in (4). For this, we consider the dynamics of the normalized state vector

$$|\psi_t\rangle = \langle\phi_t|\phi_t\rangle^{-1/2}|\phi_t\rangle. \tag{119}$$

If we write $N_t = \langle\phi_t|\phi_t\rangle^{1/2}$ for the normalization factor, then by Ito's lemma we obtain

$$\mathrm{d}N_t^{-1} = \tfrac{3}{8}\sigma^2 H_t^2 N_t^{-1}\,\mathrm{d}t - \tfrac{1}{2}\sigma H_t N_t^{-1}\,\mathrm{d}\xi_t. \tag{120}$$

Hence, for the dynamics of the normalized state $|\psi_t\rangle = N_t^{-1}|\phi_t\rangle$ we have

$$\mathrm{d}|\psi_t\rangle = N_t^{-1}\,\mathrm{d}|\phi_t\rangle + \left(\mathrm{d}N_t^{-1}\right)|\phi_t\rangle + \left(\mathrm{d}N_t^{-1}\right)(\mathrm{d}|\phi_t\rangle), \tag{121}$$

and thus

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}|\psi_t\rangle\,\mathrm{d}t - \tfrac{1}{8}\sigma^2\left(\hat{H}^2 + 2\hat{H}H_t - 3H_t^2\right)|\psi_t\rangle\,\mathrm{d}t + \tfrac{1}{2}(\hat{H} - H_t)|\psi_t\rangle\,\mathrm{d}\xi_t. \tag{122}$$

This expression can be simplified if we introduce a process $\{W_t\}$ by the relation

$$W_t = \xi_t - \sigma\int_0^t H_u\,\mathrm{d}u. \tag{123}$$

Then the dynamics for the normalized state vector $\{|\psi_t\rangle\}$ can be written as

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}|\psi_t\rangle\,\mathrm{d}t - \tfrac{1}{8}\sigma^2(\hat{H} - H_t)^2|\psi_t\rangle\,\mathrm{d}t + \tfrac{1}{2}\sigma(\hat{H} - H_t)|\psi_t\rangle\,\mathrm{d}W_t, \tag{124}$$

which is identical in form to (3), and this leaves us with the problem of the interpretation of the process $\{W_t\}$.

Now $\{\xi_t\}$ is by hypothesis a $\mathbb{Q}$-Brownian motion, so evidently $\{W_t\}$, as defined in (123), is a $\mathbb{Q}$-Brownian motion with drift. We can, however, find a new measure $\mathbb{P}$ with respect to which $\{W_t\}$ is a $\mathbb{P}$-Brownian motion. The precise statement is as follows. Let us fix a finite time $T < \infty$. Then the relevant change-of-measure density $\mathbb{Q}$-martingale $\{\Phi_t\}_{0\leqslant t\leqslant T}$ appropriate for transforming from $\mathbb{Q}$ to $\mathbb{P}$ over the time horizon $t \in [0, T]$ is defined by $\Phi_t = \langle\phi_t|\phi_t\rangle$ or equivalently (118). Thus, if $A \in \mathcal{F}_T$ denotes any $\mathcal{F}_T$-measurable set, and if $\mathbb{E}^{\mathbb{Q}}$ denotes expectation with respect to the measure $\mathbb{Q}$, then we *define* the probability of the event A with respect to the measure $\mathbb{P}$ by the formula

$$\mathbb{P}(A) = \mathbb{E}^{\mathbb{Q}}[\Phi_T \mathbf{1}_A]. \tag{125}$$

The theorem of Girsanov [20, 28, 29] allows us to infer that if $\{\xi_t\}$ is a $\mathbb{Q}$-Brownian motion, then the process $\{W_t\}_{0\leqslant t\leqslant T}$ defined by (123) is a $\mathbb{P}$-Brownian motion over the given time horizon.

We note, incidentally, that if $\{m_t\}$ is any $\mathbb{Q}$-martingale, then the process $\{M_t\}_{0\leqslant t\leqslant T}$ defined by $M_t = m_t/\Phi_t$ is a $\mathbb{P}$-martingale. In particular, since the process

$$\begin{aligned}\langle\phi_t|\hat{H}|\phi_t\rangle &= \langle\phi_0|\hat{H}\exp\left(\sigma\hat{H}\xi_t - \frac{1}{2}\sigma^2\hat{H}^2 t\right)|\phi_0\rangle \\ &= \sum_i \pi_i E_i \exp\left(\sigma E_i\xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right) \qquad (126)\end{aligned}$$

is a $\mathbb{Q}$-martingale (a sum of geometric Brownian motions is a martingale), it follows that the energy process $\{H_t\}$ defined by

$$H_t = \frac{\langle\phi_t|\hat{H}|\phi_t\rangle}{\langle\phi_t|\phi_t\rangle} = \frac{\langle\psi_t|\hat{H}|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \qquad (127)$$

is a $\mathbb{P}$-martingale. Therefore, for any finite time horizon $[0, T]$ the dynamics of (3) can be reproduced by the following procedure. First, we solve the ancillary equation (108) with the required initial condition. Next, the solution thus obtained is used to construct the processes $\{H_t\}_{0\leqslant t\leqslant T}$, $\{\Phi_t\}_{0\leqslant t\leqslant T}$ and $\{|\psi_t\rangle\}_{0\leqslant t\leqslant T}$. Finally, the change-of-measure density martingale $\{\Phi_t\}$ is used to change from the 'ancillary' measure $\mathbb{Q}$ to the 'physical' measure $\mathbb{P}$, which is used to interpret the statistical properties of the dynamics of the quantum system.

With this information at hand, we can now present another useful characterization of the dynamics of the state-vector process. We begin with (3) and (4), and introduce the process $\{\xi_t\}$ by use of the relation (123). The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\{\mathcal{F}_t\}$ are defined, with respect to which $\{W_t\}$ is a standard Brownian motion. We introduce on this probability space the state-vector process $\{|\Psi_t\rangle\}$ by writing

$$|\Psi_t\rangle = \exp\left(-\mathrm{i}\hat{H}t - \frac{1}{4t}(\xi_t - \sigma\hat{H}t)^2\right)|\psi_0\rangle. \qquad (128)$$

Then it should be evident that

$$|\psi_t\rangle = \frac{|\Psi_t\rangle}{\sqrt{\langle\Psi_t|\Psi_t\rangle}}, \qquad (129)$$

and hence that $|\Psi_t\rangle$ is an unnormalized form of the state vector $|\psi_t\rangle$. In fact, so is the state vector $|\phi_t\rangle$, but $|\Psi_t\rangle$ and $|\phi_t\rangle$ have different norms.

The significance of the process $\{|\Psi_t\rangle\}$ is that this process is identical (modulo straightforward minor changes in notation) to the nonunitary evolution introduced and used by Pearle [24, 25] for the formulation and analysis of collapse models. In particular, equation (2.1) of [25] is identical to our equation (128) above. Pearle [25] asserts that (128) represents 'the most transparent formulation of the energy-based collapse model'. Although (128) does indeed represent a formulation of the model, it can hardly be regarded as transparent. The problem is that the definition of $\{\xi_t\}$ involves $|\Psi_t\rangle$, and hence (128) is, in effect, *no more than an integral representation of the nonlinear stochastic differential equation* (3). To put the matter differently, whereas $\{|\Psi_t\rangle\}$ depends on $\{\xi_t\}$, the probability law of the process $\{\xi_t\}$ depends on $\{\langle\Psi_t|\Psi_t\rangle\}$; this is the content of equation (2.2) of [25]. Thus, when in what follows we speak of obtaining a 'solution' to (3), it should be emphasized that we are not merely seeking a 'reformulation' such as that represented by (128) or a change-of-measure induced linearization.

11. Observation of the energy in the presence of noise

We now present a general method for obtaining an explicit solution to the stochastic differential equation (3). The method we propose ties in very suggestively with the theory of nonlinear filtering as developed for example in [21].

Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given, and let $\{\mathcal{G}_t\}$ be a filtration of $\mathcal{F}$ with respect to which a standard Brownian motion $\{B_t\}$ is specified, together with an independent random variable H. We assume that H is $\mathcal{G}_0$-measurable and that it takes the values $\{E_i\}_{i=1,2,\ldots,N}$ with the probabilities $\{\pi_i\}_{i=1,2,\ldots,N}$. If we look ahead briefly to the results that will eventually follow, the random variable H will have the interpretation of representing the terminal value of the energy after state reduction, given the Hamiltonian $\hat{H}$ and the initial state $|\psi_0\rangle$. However, for the moment we assign no *a priori* physical significance to H and $\{B_t\}$, which are introduced as an ansatz for obtaining a solution for (3).

Now suppose we define a random process $\{\xi_t\}_{0\leqslant t<\infty}$ according to the scheme

$$\xi_t = \sigma H t + B_t, \tag{130}$$

where σ is a positive constant. Since our units are such that $\hbar = 1$, the random variable H can be thought of as having units of $[\mathrm{T}]^{-1}$, and hence σ, B_t and ξ_t all have units of $[\mathrm{T}]^{\frac{1}{2}}$. Later we shall identify σ with the parameter appearing in the dynamical equation (3), but for the moment we leave its value unspecified.

The process $\{\xi_t\}$ introduced here has no *a priori* connection with the process having the same name introduced in section 8. Nevertheless, as we proceed it will be indicated in what sense these processes can be identified with one another. In the probability measure $\mathbb{P}$, the process $\{\xi_t\}$ defined by (130) is a Brownian motion with a random drift rate σH. For each value of t one can think of ξ_t as providing noisy information about the random variable H. That is to say, given the value of ξ_t one can try to infer information about the value of H. The presence of the independent noise B_t interferes with this process. In particular, for small values of t, say those for which $t \ll \sigma^2$, it is typically the case that $|B_t|/\sigma t \gg 1/\sigma^2$. This follows from the fact that $\mathbb{E}[|B_t|] = \sqrt{2t/\pi}$. Thus, if $t \ll \sigma^2$ and if $|H| \ll 1/\sigma^2$, then knowledge of $\xi_t/\sigma t$ provides little information about the value of H. On the other hand, for large values of t we have $\xi_t/\sigma t \approx H$. We emphasize that at this point in our analysis the interpretation of $\{\xi_t\}$ is irrelevant, since it is being introduced as an ansatz for obtaining the solution to (3). Nevertheless, it will be worthwhile to remark as we proceed on various aspects of the nature of the 'information process' $\{\xi_t\}$.

Let $\{\mathcal{F}_t^\xi\}$ denote the filtration generated by $\{\xi_t\}$. We consider the process $\{H_t\}_{0\leqslant t\leqslant\infty}$ generated by the conditional expectation

$$H_t = \mathbb{E}\big[H\big|\mathcal{F}_t^\xi\big]. \tag{131}$$

Intuitively, conditioning with respect to the σ-algebra $\mathcal{F}_t^\xi$ means conditioning with respect to the outcome of the random trajectory $\{\xi_s\}_{0\leqslant s\leqslant t}$. Clearly, $\mathcal{F}_t^\xi \subset \mathcal{G}_t$ since knowledge of H together with $\{B_s\}_{0\leqslant s\leqslant t}$ implies knowledge of $\{\xi_s\}_{0\leqslant s\leqslant t}$, although the converse is not the case.

Proposition 7. *The conditional expectation* $\mathbb{E}\big[H\big|\mathcal{F}_t^\xi\big]$ *represents the best estimate for the value of H given the trajectory of the process $\{\xi_s\}_{0\leqslant s\leqslant t}$ from time 0 up to time t.*

Proof. Consider the problem of finding an $\mathcal{F}_t^\xi$-measurable random variable Y_t that minimizes the expected value of the squared deviation of H from Y_t, given the information $\mathcal{F}_t^\xi$. Thus, we wish to find a choice of Y_t that for each $\omega \in \Omega$ minimizes

$$J_t = \mathbb{E}\big[(H - Y_t)^2\big|\mathcal{F}_t^\xi\big]. \tag{132}$$

Since Y_t is assumed to be $\mathcal{F}_t^\xi$-measurable, we have

$$\mathbb{E}\big[(H-Y_t)^2\big|\mathcal{F}_t^\xi\big] = \mathbb{E}\big[H^2\big|\mathcal{F}_t^\xi\big] - 2Y_t\mathbb{E}\big[H\big|\mathcal{F}_t^\xi\big] + Y_t^2. \tag{133}$$

Now setting $Y_t = \mathbb{E}\big[H\big|\mathcal{F}_t^\xi\big] + Z_t$, where Z_t is any $\mathcal{F}_t^\xi$-measurable random variable, we find that

$$J_t = \mathbb{E}\big[(H-H_t)^2\big|\mathcal{F}_t^\xi\big] + Z_t^2, \tag{134}$$

where $H_t = \mathbb{E}\big[H\big|\mathcal{F}_t^\xi\big]$. Therefore J_t achieves its minimum if and only if $Z_t = 0$. □

The intuition behind this result is as follows. We can think of H as being a hidden variable. Its value is hidden by virtue of the noise process $\{B_t\}$. The best estimate available at time t for the value of H is the process $\{H_t\}$ defined by (131). Our goal now is to show that $\{H_t\}$ can be identified with the energy expectation process (4) associated with the standard energy-based stochastic extension of the Schrödinger equation.

12. Optimal estimation of the energy

We proceed in this section to calculate the conditional expectation (131) to establish the following useful result.

Proposition 8. *Let H be a random variable taking the value E_i with probability π_i $(i = 1, 2, \ldots, n)$, and set $\xi_t = \sigma Ht + B_t$ for $0 \leqslant t < \infty$, where σ is a constant and the Brownian motion $\{B_t\}$ is independent of H. Then the conditional expectation $H_t = \mathbb{E}\big[H\big|\mathcal{F}_t^\xi\big]$ is given by*

$$H_t = \frac{\sum_i \pi_i E_i \exp\big(\sigma E_i\xi_t - \frac{1}{2}\sigma^2E_i^2t\big)}{\sum_i \pi_i \exp\big(\sigma E_i\xi_t - \frac{1}{2}\sigma^2E_i^2t\big)}. \tag{135}$$

Proof. First, we observe that $\{\xi_t\}$ is a Markov process. To establish that $\{\xi_t\}$ is Markovian, we need to show that for all $T \geqslant t$ the conditional probability distribution of ξ_T given the history $\{\xi_s\}_{0\leqslant s\leqslant t}$ is equal to the conditional probability distribution of ξ_T given the value ξ_t of the process at time t alone. In other words, we need to establish the following result:

Lemma 1. *Let $\xi_t = \sigma Ht + B_t$, where H is a random variable taking the values E_i $(i = 1, 2, \ldots, N)$ with probability $\mathbb{P}(H = E_i) = \pi_i$, σ is a constant and $\{B_t\}$ is a standard $\mathbb{P}$-Brownian motion, independent of H. Then for all $T \geqslant t$ and for all $x \in \mathbb{R}$ we have*

$$\mathbb{P}\big(\xi_T \leqslant x\big|\mathcal{F}_t^\xi\big) = \mathbb{P}(\xi_T \leqslant x|\xi_t). \tag{136}$$

Proof of lemma 1. It suffices to show that

$$\mathbb{P}\big(\xi_t \leqslant x|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \xi_{s_k}\big) = \mathbb{P}(\xi_t \leqslant x|\xi_s) \tag{137}$$

for any collection of times $t, s, s_1, s_2, \ldots, s_k$ such that $t \geqslant s \geqslant s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k > 0$. Now it is a remarkable property of Brownian motion that for any times t, s, s_1 satisfying $t > s > s_1 > 0$ one can show that

$$B_t \quad \text{and} \quad \frac{B_s}{s} - \frac{B_{s_1}}{s_1} \quad \text{are independent.} \tag{138}$$

More generally, if $s > s_1 > s_2 > s_3 > 0$, we find that

$$\frac{B_s}{s} - \frac{B_{s_1}}{s_1} \quad \text{and} \quad \frac{B_{s_2}}{s_2} - \frac{B_{s_3}}{s_3} \quad \text{are independent.} \tag{139}$$

In each case the result stated follows after a calculation of the covariance of the indicated variables. Next we note that

$$\frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1} = \frac{B_s}{s} - \frac{B_{s_1}}{s_1}. \tag{140}$$

It follows therefore that

$$\begin{aligned}\mathbb{P}\big(\xi_t \leqslant x|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \xi_{s_k}\big) &= \mathbb{P}\left(\xi_t \leqslant x \,\middle|\, \xi_s, \frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1}, \frac{\xi_{s_1}}{s_1} - \frac{\xi_{s_2}}{s_2}, \ldots, \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k}\right)\\ &= \mathbb{P}\left(\xi_t \leqslant x \,\middle|\, \xi_s, \frac{B_s}{s} - \frac{B_{s_1}}{s_1}, \frac{B_{s_1}}{s_1} - \frac{B_{s_2}}{s_2}, \ldots, \frac{B_{s_{k-1}}}{s_{k-1}} - \frac{B_{s_k}}{s_k}\right).\end{aligned} \tag{141}$$

However, since ξ_t and ξ_s are independent of $B_s/s - B_{s_1}\big/s_1$, $B_{s_1}\big/s_1 - B_{s_2}\big/s_2, \ldots, B_{s_{k-1}}\big/s_{k-1} - B_{s_k}\big/s_k$, the desired result of lemma 1 follows. □

Continuing with the proof of Proposition 8, we note next that

$$\mathbb{E}\big[H\big|\mathcal{F}_t^{\xi}\big] = \mathbb{E}[H|\xi_t]. \tag{142}$$

That is to say, rather than conditioning on the σ-subalgebra $\mathcal{F}_t^{\xi}$ generated by $\{\xi_s\}_{0\leqslant s\leqslant t}$ it suffices to condition on ξ_t alone (conditioning with respect to a random variable means conditioning with respect to the σ-algebra generated by that random variable). The additional information in $\{\xi_s\}_{0\leqslant s\leqslant t}$ does not allow us to improve the estimate of H once ξ_t has been given. This follows from the fact that $\{\xi_t\}$ is Markovian and that

$$\lim_{t\to\infty} \frac{\xi_t}{t} = \sigma H. \tag{143}$$

To calculate $\mathbb{E}[H|\xi_t]$, we require a version of the Bayes formula applicable when we consider the probability of a discrete random variable conditioned on the value of a continuous random variable. In this connection, we recall for convenience that for discrete random variables A and B that take on the values $A = A_i$ $(i = 1, 2, \ldots, n)$ and $B = B_j$ $(j = 1, 2, \ldots, m)$ with probabilities q_i and r_j, respectively, then we have the classical Bayes formula

$$\mathbb{P}(A = A_i|B = B_j) = \frac{\mathbb{P}(A = A_i)\mathbb{P}(B = B_j|A = A_i)}{\mathbb{P}(B = B_j)}, \tag{144}$$

or equivalently

$$\mathbb{P}(A = A_i|B = B_j) = \frac{q_i\mathbb{P}(B = B_j|A = A_i)}{\sum_{i=1}^n q_i\mathbb{P}(B = B_j|A = A_i)}, \tag{145}$$

since the marginal probability for the random variable B can be written as

$$\mathbb{P}(B = B_j) = \sum_{i=1}^n q_i\mathbb{P}(B = B_j|A = A_i). \tag{146}$$

Alternatively, instead of conditioning directly with respect to the event $B = B_j$ we can condition with respect to the random variable B, and write

$$\begin{aligned}\mathbb{P}(A = A_i|B) &= \frac{\mathbb{P}(A = A_i)\mathbb{P}(B|A = A_i)}{\mathbb{P}(B)}\\ &= \frac{q_i\mathbb{P}(B|A = A_i)}{\sum_{i=1}^n q_i\mathbb{P}(B|A = A_i)},\end{aligned} \tag{147}$$

where $\mathbb{P}(B)$ is the random variable that takes the value $r_j = \mathbb{P}(B = B_j)$ when B takes value B_j, and $\mathbb{P}(B|A = A_i)$ is the random variable that takes the value $\mathbb{P}(B = B_j|A = A_i)$ when B takes value B_j. Clearly,

$$\mathbb{P}(B) = \sum_{i=1}^{n} q_i \mathbb{P}(B|A = A_i). \tag{148}$$

For our purpose we need the analogue of (147) applicable in the situation where A is a discrete random variable and B is a continuous random variable. In that case

$$\mathbb{P}(A = A_i|B) = \frac{\mathbb{P}(A = A_i)\rho(B|A = A_i)}{\rho(B)}, \tag{149}$$

or equivalently

$$\mathbb{P}(A = A_i|B) = \frac{q_i\rho(B|A = A_i)}{\sum_{i=1}^{n} q_i\rho(B|A = A_i)}, \tag{150}$$

since

$$\rho(B) = \sum_{i=1}^{n} q_i \rho(B|A = A_i). \tag{151}$$

Here $\rho(x)$ denotes the density function of the continuous random variable B, so

$$\mathbb{P}(B < b) = \int_{-\infty}^{b} \rho(x)\,\mathrm{d}x, \tag{152}$$

and $\rho(x|A = A_i)$ is the conditional density of B given $A = A_i$, so

$$\mathbb{P}(B < b|A = A_i) = \int_{-\infty}^{b} \rho(x|A = A_i)\,\mathrm{d}x. \tag{153}$$

The random variable $\rho(B)$, *resp.* $\rho(B|A = A_i)$, takes the value $\rho(b)$, *resp.* $\rho(b|A = A_i)$, when B takes the value b.

Equation (150) is the version of the Bayes formula we require in order to determine the conditional expectation (131). In particular, since ξ_t is a continuous random variable, we have

$$\begin{aligned}\mathbb{P}(H = E_i|\xi_t) &= \frac{\mathbb{P}(H = E_i)\rho(\xi_t|H = E_i)}{\rho(\xi_t)} \\ &= \frac{\mathbb{P}(H = E_i)\rho(\xi_t|H = E_i)}{\sum_i \mathbb{P}(H = E_i)\rho(\xi_t|H = E_i)} \\ &= \frac{\pi_i\rho(\xi_t|H = E_i)}{\sum_i \pi_i\rho(\xi_t|H = E_i)}.\end{aligned} \tag{154}$$

Here $\rho(\xi_t|H = E_i)$ denotes the conditional density function for the random variable ξ_t given that $H = E_i$. Since $\{B_t\}$ is a standard Brownian motion in the $\mathbb{P}$-measure, the conditional probability density for ξ_t is Gaussian and is given by

$$\rho(\xi_t|H = E_i) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(\xi_t - \sigma E_i t)^2\right). \tag{155}$$

It follows from the Bayes law (154) that the desired conditional probability for the random variable H is given by

$$\mathbb{P}(H = E_i|\xi_t) = \frac{\pi_i \exp\left(\sigma E_i\xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\sum_i \pi_i \exp\left(\sigma E_i\xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}. \tag{156}$$

Therefore we deduce that

$$\begin{aligned} H_t &= \mathbb{E}[H|\xi_t] = \sum_i E_i \mathbb{P}(H = E_i|\xi_t) \\ &= \frac{\sum_i \pi_i E_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\sum_i \pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}. \end{aligned} \tag{157}$$

That concludes the proof of proposition 7. □

More generally (see, e.g., [33]), a similar argument establishes that for any bounded function $x \to f(x)$ we have

$$\mathbb{E}\left[f(H)\middle|\mathcal{F}_t^\xi\right] = \frac{\sum_i \pi_i f(E_i) \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}{\sum_i \pi_i \exp\left(\sigma E_i \xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)}. \tag{158}$$

13. Existence of the innovation process

We now proceed to establish the following basic result.

Proposition 9. *Let $\{\xi_t\}$ and $\{H_t\}$ be defined as in* Proposition 8. *Then the process $\{W_t\}$ defined by*

$$W_t = \xi_t - \sigma \int_0^t H_s \, \mathrm{d}s \tag{159}$$

is an $\left\{\mathcal{F}_t^\xi\right\}$-Brownian motion.

Proof. Starting with the relation $\xi_t = \sigma H t + B_t$ we define $\{W_t\}$ as above, with $\{H_t\}$ defined as in (131). To show that $\{W_t\}$ is an $\left\{\mathcal{F}_t^\xi\right\}$-Brownian motion it suffices to show that $\{W_t\}$ is an $\left\{\mathcal{F}_t^\xi\right\}$-martingale and that $(\mathrm{d}W_t)^2 = \mathrm{d}t$. First, we shall demonstrate that $\{W_t\}$ is an $\left\{\mathcal{F}_t^\xi\right\}$-martingale. Letting $t \leqslant T$ we deduce that

$$\begin{aligned} \mathbb{E}\left[W_T\middle|\mathcal{F}_t^\xi\right] &= \mathbb{E}\left[\xi_T\middle|\mathcal{F}_t^\xi\right] - \sigma\mathbb{E}\left[\int_0^T H_s \, \mathrm{d}s\middle|\mathcal{F}_t^\xi\right] \\ &= \sigma T \mathbb{E}\left[H\middle|\mathcal{F}_t^\xi\right] + \mathbb{E}\left[B_T\middle|\mathcal{F}_t^\xi\right] - \sigma\mathbb{E}\left[\int_0^T H_s \, \mathrm{d}s\middle|\mathcal{F}_t^\xi\right] \\ &= \sigma T \mathbb{E}\left[H\middle|\mathcal{F}_t^\xi\right] + \mathbb{E}\left[B_T\middle|\mathcal{F}_t^\xi\right] - \sigma \int_0^T \mathbb{E}\left[H_s\middle|\mathcal{F}_t^\xi\right] \mathrm{d}s, \end{aligned} \tag{160}$$

by Fubini's theorem. Next, we note that

$$\begin{aligned} \int_0^T \mathbb{E}\left[H_s\middle|\mathcal{F}_t^\xi\right] \mathrm{d}s &= \int_0^t \mathbb{E}\left[H_s\middle|\mathcal{F}_t^\xi\right] \mathrm{d}s + \int_t^T \mathbb{E}\left[H_s\middle|\mathcal{F}_t^\xi\right] \mathrm{d}s \\ &= \int_0^t H_s \, \mathrm{d}s + \int_t^T H_t \, \mathrm{d}s \\ &= \int_0^t H_s \, \mathrm{d}s + (T - t) H_t. \end{aligned} \tag{161}$$

Here we have used the fact that $\{H_t\}$ is an $\left\{\mathcal{F}_t^\xi\right\}$-martingale. Substituting (161) into (160) we obtain

$$\mathbb{E}\left[W_T\middle|\mathcal{F}_t^\xi\right] = \sigma t \mathbb{E}\left[H\middle|\mathcal{F}_t^\xi\right] + \mathbb{E}\left[B_T\middle|\mathcal{F}_t^\xi\right] - \sigma \int_0^t H_s \, \mathrm{d}s. \tag{162}$$

Finally, we observe that by the tower property of conditional expectation we have

$$\mathbb{E}\big[B_T\big|\mathcal{F}_t^{\xi}\big] = \mathbb{E}\big[\mathbb{E}\big[B_T\big|\mathcal{F}_t^{B}, H\big]\big|\mathcal{F}_t^{\xi}\big] = \mathbb{E}\big[B_t\big|\mathcal{F}_t^{\xi}\big]. \tag{163}$$

Inserting this into (162) we obtain

$$\begin{aligned}\mathbb{E}\big[W_T\big|\mathcal{F}_t^{\xi}\big] &= \sigma t H_t + \mathbb{E}\big[B_t\big|\mathcal{F}_t^{\xi}\big] - \sigma\int_0^t H_s\,\mathrm{d}s \\ &= \mathbb{E}\big[(\sigma t H + B_t)\big|\mathcal{F}_t^{\xi}\big] - \sigma\int_0^t H_s\,\mathrm{d}s \\ &= \mathbb{E}\big[\xi_t\big|\mathcal{F}_t^{\xi}\big] - \sigma\int_0^t H_s\,\mathrm{d}s = W_t,\end{aligned} \tag{164}$$

and this establishes that $\{W_t\}$ is an $\{\mathcal{F}_t^{\xi}\}$-martingale. Next, we observe that since

$$\mathrm{d}W_t = \sigma(H - H_t)\,\mathrm{d}t + \mathrm{d}B_t, \tag{165}$$

it follows at once that $(\mathrm{d}W_t)^2 = \mathrm{d}t$. This, together with the fact that $\{W_t\}$ is an $\{\mathcal{F}_t^{\xi}\}$-martingale, implies that $\{W_t\}$ is an $\{\mathcal{F}_t^{\xi}\}$-Brownian motion. □

We call $\{W_t\}$ the *innovation process* associated with the dynamics of the wavefunction. The significance of the fact that $\{W_t\}$ is an $\{\mathcal{F}_t^{\xi}\}$-Brownian motion is that the process $\{\xi_t\}$ as defined in (130) satisfies a diffusion equation of the form

$$\mathrm{d}\xi_t = \sigma H_t\,\mathrm{d}t + \mathrm{d}W_t, \tag{166}$$

where $H_t = H(\xi_t, t)$. As a result, one can prove that $\mathcal{F}_t^{\xi} = \mathcal{F}_t^{W}$; that is to say, the information set generated by $\{W_t\}$ is equivalent to that generated by $\{\xi_t\}$. It is the innovation process $\{W_t\}$, and not the noise $\{B_t\}$, that 'drives' the dynamics of the state-vector process $\{|\psi_t\rangle\}$ in (3).

Now let $|\psi_0\rangle$ be the initial normalized state vector of the quantum system, and let $\hat{\Pi}_i$ denote for each value of i the projection operator onto the subspace of Hilbert space corresponding to the energy eigenvalue E_i, which may be degenerate. As before, we let

$$|\phi_i\rangle = \pi_i^{-1/2}\hat{\Pi}_i|\psi_0\rangle \tag{167}$$

denote the Lüders state corresponding to E_i, and we write

$$\pi_{it} = \mathbb{P}(H = E_i|\xi_t) \tag{168}$$

for the process defined by (156).

Theorem 1. *The solution of the stochastic differential equation*

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}|\psi_t\rangle\,\mathrm{d}t - \tfrac{1}{8}\sigma^2(\hat{H} - H_t)^2|\psi_t\rangle\,\mathrm{d}t + \tfrac{1}{2}\sigma(\hat{H} - H_t)|\psi_t\rangle\,\mathrm{d}W_t \tag{169}$$

with initial condition $|\psi_0\rangle$ *is given by*

$$|\psi_t\rangle = \sum_i \mathrm{e}^{-\mathrm{i}E_i t}\pi_{it}^{1/2}|\phi_i\rangle. \tag{170}$$

Here, $|\phi_i\rangle$ *denotes the Lüders state for the eigenvalue* E_i*, and*

$$\pi_{it} = \frac{\pi_i\exp\big(\sigma E_i\xi_t - \tfrac{1}{2}\sigma^2E_i^2t\big)}{\sum_i\pi_i\exp\big(\sigma E_i\xi_t - \tfrac{1}{2}\sigma^2E_i^2t\big)}, \tag{171}$$

where $\xi_t = \sigma H t + B_t$*. The random variable* H *takes the value* $\{E_i\}$ *with the probabilities* $\{\pi_i\}$*, and* $\{B_t\}$ *is a Brownian motion independent of* H*. The process* $\{H_t\}$ *is defined by* $H_t = \sum_i E_i\pi_{it}$ *and the* $\{\mathcal{F}_t^{\xi}\}$*-Brownian motion* $\{W_t\}$ *is given by* $W_t = \xi_t - \sigma\int_0^t H_u\,\mathrm{d}u$.

Proof. It is a straightforward exercise to verify that (170) satisfies the stochastic differential equation (3) with the given initial condition. In particular, by applying Ito's lemma to (171) and using the relation (166), we can verify that $\{\pi_{it}\}$ satisfies

$$\mathrm{d}\pi_{it} = \sigma(E_i - H_t)\pi_{it}\,\mathrm{d}W_t. \tag{172}$$

Then with another application of Ito's lemma we deduce that

$$\mathrm{d}\pi_{it}^{1/2} = -\tfrac{1}{8}\sigma^2(E_i - H_t)^2\pi_{it}^{1/2}\,\mathrm{d}t + \tfrac{1}{2}\sigma(E_i - H_t)\pi_{it}^{1/2}\,\mathrm{d}W_t, \tag{173}$$

and with this relation at hand a short calculation shows that (170) satisfies (3). □

14. Direct verification of the reductive property

Thus, summing up, the stochastic equation (3) can be solved as follows. We let H be a random variable taking values $\{E_i\}$ with the probabilities $\{\pi_i\}$ defined by (64) or equivalently by

$$\pi_i = \frac{\langle\psi_0|\hat{\Pi}_i|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}. \tag{174}$$

Letting $\{B_t\}$ denote an independent Brownian motion, we define the process $\{\xi_t\}_{0\leqslant t<\infty}$ by writing $\xi_t = \sigma H t + B_t$. The solution of (3) is then given by (170) or equivalently

$$|\psi_t\rangle = \frac{\sum_i \pi_i^{1/2}\exp\left(-\mathrm{i}E_i t + \frac{1}{2}\sigma E_i\xi_t - \frac{1}{4}\sigma^2 E_i^2 t\right)|\phi_i\rangle}{\left(\sum_i \pi_i \exp\left(\sigma E_i\xi_t - \frac{1}{2}\sigma^2 E_i^2 t\right)\right)^{1/2}}, \tag{175}$$

where the $\{\mathcal{F}_t^{\xi}\}$-Brownian motion $\{W_t\}$ driving $\{|\psi_t\rangle\}$ in (169) is given by (159), with $\{H_t\}$ defined as in (157).

The fact that (157) defines a reduction process for the energy can be verified directly as follows. Suppose, in a particular realization of the process $\{H_t\}$, the random variable H takes the value E_j for some specific choice of the index j. That is to say, we condition on the event $H = E_j$. Substituting $\xi_t = \sigma E_j t + B_t$ for the corresponding realization of $\{H_t\}$, we have

$$\begin{aligned} H_t &= \frac{\sum_i \pi_i E_i \exp\left(\sigma E_i B_t - \frac{1}{2}\sigma^2 E_i(E_i - 2E_j)t\right)}{\sum_i \pi_i \exp\left(\sigma E_i B_t - \frac{1}{2}\sigma^2 E_i(E_i - E_j)t\right)} \\ &= \frac{\sum_i \pi_i E_i \exp\left(\sigma(E_i - E_j)B_t - \frac{1}{2}\sigma^2(E_i - E_j)^2 t\right)}{\sum_i \pi_i \exp\left(\sigma(E_i - E_j)B_t - \frac{1}{2}\sigma^2(E_i - E_j)^2 t\right)} \\ &= \frac{\pi_j E_j + \sum_{i(\neq j)} \pi_i E_i \exp\left(\sigma(E_i - E_j)B_t - \frac{1}{2}\sigma^2(E_i - E_j)^2 t\right)}{\pi_j + \sum_{i(\neq j)} \pi_i \exp\left(\sigma(E_i - E_j)B_t - \frac{1}{2}\sigma^2(E_i - E_j)^2 t\right)}. \end{aligned} \tag{176}$$

However, the martingale $\{M_{ijt}\}$ defined for $i \neq j$ by

$$M_{ijt} = \exp\left(\sigma(E_i - E_j)B_t - \tfrac{1}{2}\sigma^2(E_i - E_j)^2 t\right), \tag{177}$$

which appears in (176), has the following property:

$$\lim_{t\to\infty}\mathbb{P}(M_{ijt} > 0) = 0. \tag{178}$$

In other words, $\{M_{ijt}\}$ converges to zero for large t with probability 1. We note, incidentally, that a geometric Brownian motion, i.e. a process of the form

$$X_t = \exp\left(\nu B_t - \tfrac{1}{2}\nu^2 t\right), \tag{179}$$

has the property that it converges to unity in expectation but to zero in probability. That is to say, $\lim_{t\to\infty}\mathbb{E}[X_t] = 1$ whereas $\lim_{t\to\infty}\mathbb{P}(X_t > 0) = 0$. Since

$$H_t = \frac{\pi_j E_j + \sum_{i(\neq j)} \pi_i E_i M_{ijt}}{\pi_j + \sum_{i(\neq j)} \pi_i M_{ijt}}, \tag{180}$$

we see that $\{H_t\}$ converges to the value E_j with probability 1. A similar argument immediately shows that if $H = E_j$ then for each value of i we have

$$\lim_{t\to\infty} \pi_{it} = \mathbf{1}_{\{i=j\}}, \tag{181}$$

which shows that $\{|\psi_t\rangle\}$ converges to the Lüders state corresponding to the energy eigenvalue j with probability 1, in accordance with the results noted in [4].

The advantage of the expression (157) is that $\{H_t\}$ and $\{|\psi_t\rangle\}$ are expressed *algebraically* in terms of the underlying random variable H and the independent Brownian motion $\{B_t\}$. As a consequence, we can directly investigate and verify various properties of the reduction process (3) without having to resort to numerical integration.

15. Identification of independent noise and energy

In solving the stochastic equation (3) we have introduced in section 11 the idea of filtering, that is, estimation of the value of the random variable H, given noisy information about H, where the noise is induced by an independent random process $\{B_t\}$. Although the method is useful in obtaining an analytical solution to (3), the introduction of these random variables might appear artificial, because it is not immediately obvious how these variables emerge out of the problem specified by (3). Remarkably, however, it turns out that we can *derive* the quantities introduced in section 11 from the ingredients specified in (3) and (4). The aim of this section is to show how this can be achieved. We start with the following result.

Proposition 10. *Let $\{H_t\}$ denote the process defined by (4) and $\{\xi_t\}$ the process defined by (91). The random variables $H_\infty = \lim_{s\to\infty} H_s$ and $B_t = \xi_t - \sigma t H_\infty$ are independent for all t. Furthermore, the process $\{B_t\}$ is a standard Brownian motion.*

Proof. We begin by establishing the independence of the random variables B_t and H_∞. To this end, we note that it suffices to verify that the relation

$$\mathbb{E}[\mathrm{e}^{xB_t+yH_\infty}] = \mathbb{E}[\mathrm{e}^{xB_t}]\mathbb{E}[\mathrm{e}^{yH_\infty}] \tag{182}$$

holds for all x, y. The verification of this property proceeds as follows. Using the tower property of conditional expectation (see section 2) we have

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{xB_t+yH_\infty}] &= \mathbb{E}\big[\mathbb{E}\big[\mathrm{e}^{xB_t+yH_\infty}\big|\mathcal{F}_t^W\big]\big] \\ &= \mathbb{E}\big[\mathbb{E}\big[\mathrm{e}^{x\xi_t+(y-\sigma xt)H_\infty}\big|\mathcal{F}_t^W\big]\big] \\ &= \mathbb{E}\big[\mathrm{e}^{x\xi_t}\mathbb{E}\big[\mathrm{e}^{(y-\sigma xt)H_\infty}\big|\mathcal{F}_t^W\big]\big],\end{aligned} \tag{183}$$

where we have used the $\{\mathcal{F}_t^W\}$-measurability of the random variable ξ_t in the last step. Let us now consider the conditional expectation $\mathbb{E}\big[\mathrm{e}^{(y-\sigma xt)H_\infty}\big|\mathcal{F}_t^W\big]$ appearing inside the brackets in (183). By use of the expression for the conditional probability of H_∞ obtained in (156) we deduce that

$$\mathbb{E}\big[\mathrm{e}^{(y-\sigma xt)H_\infty}\big|\mathcal{F}_t^W\big] = \frac{\sum_i \pi_i \exp\big((y-\sigma xt)E_i + \xi_t E_i\sigma - \frac{1}{2}E_i^2\sigma^2 t\big)}{\sum_i \pi_i \exp\big(\xi_t E_i\sigma - \frac{1}{2}E_i^2\sigma^2 t\big)}. \tag{184}$$

To proceed further in determining the outer expectation in (183) we make use of the following subsidiary result.

Lemma 2.

$$\sum_i \pi_i \exp\left(\xi_t E_i\sigma - \frac{1}{2}E_i^2\sigma^2 t\right) = \exp\left(\sigma\int_0^t H_s\,\mathrm{d}\xi_s - \frac{1}{2}\sigma^2\int_0^t H_s^2\,\mathrm{d}s\right). \tag{185}$$

Proof. Let us write $\Phi_t = \Phi(\xi_t, t)$ for the left-hand side of (185), where $\Phi(\xi, t)$ is the function of two variables defined by

$$\Phi(\xi, t) = \sum_i \pi_i \exp\left(\xi E_i \sigma - \frac{1}{2} E_i^2 \sigma^2 t\right). \tag{186}$$

Then by Ito's lemma we have

$$\mathrm{d}\Phi_t = \dot{\Phi}_t\,\mathrm{d}t + \Phi_t'\,\mathrm{d}\xi_t + \tfrac{1}{2}\Phi_t''(\mathrm{d}\xi_t)^2, \tag{187}$$

where the dot and the prime denote differentiation with respect to t and ξ, respectively. Next we observe that $(\mathrm{d}\xi_t)^2 = \mathrm{d}t$, that $\dot{\Phi}_t + \frac{1}{2}\Phi_t'' = 0$, and that $\Phi_t' = \sigma H_t \Phi_t$, the last relation following from (157). As a consequence, we see that $\{\Phi_t\}$ satisfies

$$\mathrm{d}\Phi_t = \sigma H_t \Phi_t\,\mathrm{d}\xi_t. \tag{188}$$

Finally, we note that the integral representation for this stochastic differential equation, with initial condition $\Phi_0 = 1$, is given by the right-hand side of (185). □

The key point is that the right-hand side of (185) can be used as change-of-measure density. Recall that $\{W_t\}$ is a standard Brownian motion in the measure $\mathbb{P}$. Since $\{H_t\}$ is bounded and $\{\mathcal{F}_t^W\}$-adapted, it follows by Girsanov's theorem that there exists an equivalent probability measure $\mathbb{Q}$ such that the process $\{\xi_t\}$ defined by

$$\xi_t = W_t + \sigma \int_0^t H_s\,\mathrm{d}s \tag{189}$$

is a standard Brownian motion in the $\mathbb{Q}$-measure. We let Φ_t denote the change-of-measure density in the right-hand side of (185). Then for any $\{\mathcal{F}_t^W\}$-measurable random variable X_t the conditional expectations in these two probability measures are related according to the scheme

$$\mathbb{E}_s^{\mathbb{P}}[X_t] = \frac{1}{\Phi_s}\mathbb{E}_s^{\mathbb{Q}}[\Phi_t X_t] \qquad \text{and} \qquad \mathbb{E}_s^{\mathbb{Q}}[X_t] = \Phi_s \mathbb{E}_s^{\mathbb{P}}\left[\frac{1}{\Phi_t} X_t\right]. \tag{190}$$

Equipped with these results we proceed to determine the conditional expectation (183). In particular, if we substitute (184) into (183) and use the fact that the denominator appearing in the expectation is the change-of-measure density Φ_t, and hence $\{\xi_t\}$ is a standard Brownian motion in the $\mathbb{Q}$-measure, we can apply the second identity in (190) to deduce that

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{xB_t + yH_\infty}] &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\xi_t} \sum_i \pi_i \exp\left((y - \sigma x t)E_i + \xi_t E_i \sigma - \frac{1}{2}E_i^2\sigma^2 t\right)\right] \\
&= \sum_i \pi_i \exp\left((y - \sigma x t)E_i - \frac{1}{2}E_i^2\sigma^2 t\right)\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{(x + E_i\sigma)\xi_t}] \\
&= \sum_i \pi_i \exp\left((y - \sigma x t)E_i - \frac{1}{2}E_i^2\sigma^2 t\right)\mathrm{e}^{\frac{1}{2}(x + E_i\sigma)^2 t} \\
&= \left(\sum_i \pi_i\,\mathrm{e}^{yE_i}\right)\mathrm{e}^{\frac{1}{2}x^2 t}.
\end{aligned} \tag{191}$$

This establishes the relation (182), and hence that random variables B_t and H_∞ are independent. In addition, as a bonus the result

$$\mathbb{E}[\mathrm{e}^{xB_t}] = \mathrm{e}^{\frac{1}{2}x^2 t} \tag{192}$$

shows that B_t is normally distributed with mean zero and variance t.

To complete the proof that $\{B_t\}$ is a standard Brownian motion we are required, in addition to establishing its normality, to verify that the process $\{B_t\}$ has independent increments. Alternatively, it suffices to demonstrate that

$$\mathbb{E}[\mathrm{e}^{xB_t+y(B_T-B_t)}] = \mathbb{E}[\mathrm{e}^{xB_t}]\mathbb{E}[\mathrm{e}^{y(B_T-B_t)}] \tag{193}$$

for any nonzero constants x, y. Using the definition for $\{B_t\}$ and the tower property of conditional expectation we can write

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{xB_t+y(B_T-B_t)}] &= \mathbb{E}[\mathrm{e}^{(x-y)\xi_t+y\xi_T}\,\mathrm{e}^{-(x\sigma t+y\sigma(T-t))H_\infty}] \\ &= \mathbb{E}\big[\mathrm{e}^{(x-y)\xi_t+y\xi_T}\,\mathbb{E}\big[\mathrm{e}^{-(x\sigma t+y\sigma(T-t))H_\infty}\big|\mathcal{F}_T^W\big]\big].\end{aligned} \tag{194}$$

Once again from (156) we have

$$\mathbb{E}\big[\mathrm{e}^{-(x\sigma t+y\sigma(T-t))H_\infty}\big|\mathcal{F}_T^W\big] = \frac{\sum_i \pi_i \exp\big(-x\sigma t E_i - y\sigma(T-t)E_i + \xi_T E_i\sigma - \frac{1}{2}E_i^2\sigma^2 T\big)}{\sum_i \pi_i \exp\big(\xi_T E_i\sigma - \frac{1}{2}E_i^2\sigma^2 T\big)} \tag{195}$$

for the inner expectation in (194). Substituting (195) into (194) and noting the fact that the denominator in the expectation is the change-of-measure density Φ_T we deduce, after some rearrangement of terms, that

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{xB_t+y(B_T-B_t)}] &= \sum_i \pi_i \exp\left(-x\sigma t E_i - y\sigma(T-t)E_i - \frac{1}{2}E_i^2\sigma^2 T\right)\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{(x-y)\xi_t+(y+E_i\sigma)\xi_T}] \\ &= \exp\left(\frac{1}{2}x^2 t + \frac{1}{2}y^2(T-t)\right).\end{aligned} \tag{196}$$

Here, we have made use of the Gaussian property

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{a\xi_t+b\xi_T}] &= \exp\left(\tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}[(a\xi_t+b\xi_T)^2]\right) \\ &= \exp\left(\tfrac{1}{2}(a^2+2ab)t + \tfrac{1}{2}b^2 T\right)\end{aligned} \tag{197}$$

satisfied by the random variables ξ_t and ξ_T in the $\mathbb{Q}$-measure. The result of (196) establishes (193), and thus we conclude that the process $\{B_t\}$ is normally distributed with zero mean and variance t, and has independent increments. Therefore $\{B_t\}$ is a standard Brownian motion.

□

16. Finite-time collapse model

In the foregoing sections we have investigated the properties of energy-based collapse models for which state reduction is achieved *asymptotically* in time. That is to say, although for a suitable choice of the parameter σ the state reaches the close vicinity of one of the energy eigenstates virtually instantaneously, for a strict collapse to a state for which the energy variance vanishes identically, we must take the limit $t \to \infty$. There are circumstances, however, in which it might be preferable to formulate a model that achieves strict collapse in finite time duration. An example for such a model has been proposed recently [13]. In what follows we shall apply the methodologies developed above to work out the properties of finite-time collapse models.

The model that we consider here, which gives rise to a finite-time collapse, is given by the following stochastic equation:

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}|\psi_t\rangle\,\mathrm{d}t - \frac{1}{8}\left(\frac{\sigma T}{T-t}\right)^2(\hat{H}-H_t)^2|\psi_t\rangle\,\mathrm{d}t + \frac{1}{2}\frac{\sigma T}{T-t}(\hat{H}-H_t)|\psi_t\rangle\,\mathrm{d}W_t. \tag{198}$$

We deduce immediately from the discussion in section 7 that the dynamical law (198) preserves the norm $\langle\psi_t|\psi_t\rangle$ of the state and that the associated energy process $\{H_t\}$ is a martingale. In particular, a short calculation making use of the Ito calculus shows that the energy process satisfies

$$\mathrm{d}H_t = \sigma_t V_t \,\mathrm{d}W_t, \tag{199}$$

where we have defined, for convenience, the deterministic function $\{\sigma_t\}$ by

$$\sigma_t = \frac{\sigma T}{T-t}, \tag{200}$$

and $\{V_t\}$ is the associated variance process. Note that (198) can be obtained from (3) by the substitution $\sigma \to \sigma_t$. Thus (198) contains two freely specifiable parameters, namely, σ and T. The latter will be identified with the time at which the collapse is completed. More generally, we may regard the collapse time T as a random variable having some density $p(T)$ defined on the positive real line. Then the collapse time T itself becomes random; the analysis of this case will be pursued elsewhere. Here, we shall treat T as a fixed parameter.

In order to solve (198), we shall make an ansatz analogous to the one introduced in (130). Now from the point of view of filtering theory, the collapse of the state in the model (3) takes place only asymptotically because the 'noise-to-signal' ratio, whose magnitude is of order $\sqrt{t}/t$, vanishes only asymptotically as $t \to \infty$. Therefore, in order to achieve a finite-time collapse we consider the use of a *Brownian bridge* as the source for the noise. A Brownian bridge with duration T can be regarded as a standard Brownian motion constrained to take value zero at time $t = 0$ and also at time $t = T$. By using a Brownian bridge as the source for the noise, the value of the unknown random variable H will be revealed in finite time T, since the contribution of noise vanishes at that time. Specifically, the magnitude of noise-to-signal ratio is given by $\sqrt{(T-t)/tT}$, which vanishes as $t \to T$. It remains to be shown that the solution of such a filtering problem corresponds to the solution of a finite-time collapse model (198). In what follows we shall demonstrate that this is the case.

We thus consider the information process $\{\xi_t\}_{0\leqslant t\leqslant T}$ defined in this case by

$$\xi_t = \sigma t H + B_t - \frac{t}{T} B_T, \tag{201}$$

where σ is a constant, $\{B_t\}$ is a standard Brownian motion, and H is a discrete random variable taking the values $\{E_i\}$ with probability $\{\pi_i\}$. It is evident from definition that $\xi_0 = 0$ and that $\xi_t/\sigma t = H$. The process $\{\beta_t\}_{0\leqslant t\leqslant T}$ defined by the combination

$$\beta_t = B_t - \frac{t}{T} B_T \tag{202}$$

is a standard Brownian bridge on the interval $t \in [0, T]$ satisfying $\beta_0 = 0$ and $\beta_T = 0$. We assume that H and $\{\beta_t\}$ are independent. It should be evident from the definition (202) that a Brownian bridge is normally distributed with mean $\mathbb{E}[\beta_t] = 0$ and covariance

$$\begin{aligned}\mathrm{Cov}[\beta_s, \beta_t] &= \mathbb{E}\left[B_s B_t - \frac{1}{T}(sB_t + tB_s)B_T + \frac{1}{T^2} st B_T^2\right] \\ &= s\left(1 - \frac{t}{T}\right)\end{aligned} \tag{203}$$

for $s \leqslant t$. In deriving (203) we have made use of the independent increments property satisfied by $\{B_t\}$ to deduce that $\mathbb{E}[B_s B_t] = \mathbb{E}[B_s(B_t - B_s + B_s)] = s$. The Brownian bridge, on the other hand, does not possess independent increments.

Our objective now, as before, is to determine the best estimate for the variable H given the information concerning the trajectory $\{\xi_u\}_{0\leqslant u\leqslant t}$ of the process $\{\xi_u\}$ from time $u = 0$ to time

$u = t \leqslant T$. In particular, the conclusion of proposition 7 remains valid in the present context: that is to say, the estimate that minimizes the quadratic error is given by the conditional expectation $\mathbb{E}\big[H\big|\mathcal{F}_t^{\xi}\big]$. To calculate this conditional expectation we shall make use of the following key result:

Lemma 3. *Let $\xi_t = \sigma t H + \beta_t$, where H is a random variable taking the values E_i $(i = 1, 2, \ldots, N)$ with probability $\mathbb{P}(H = E_i) = \pi_i$, σ is a constant and $\{\beta_t\}$ is a standard $\mathbb{P}$-Brownian bridge on the interval $t \in [0, T]$, independent of H. Then $\{\xi_t\}_{0\leqslant t\leqslant T}$ is a Markov process.*

Proof of lemma 3. To show that $\{\xi_t\}$ is Markovian, we must show that

$$\mathbb{P}\big(\xi_t \leqslant x\big|\mathcal{F}_s^{\xi}\big) = \mathbb{P}(\xi_t \leqslant x|\xi_s) \tag{204}$$

for all $x \in \mathbb{R}$ and all s, t such that $0 \leqslant s \leqslant t \leqslant T$. It will suffice to verify that

$$\mathbb{P}\big(\xi_t \leqslant x\big|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \xi_{s_k}\big) = \mathbb{P}(\xi_t \leqslant x|\xi_s) \tag{205}$$

for any times $t, s, s_1, s_2, \ldots, s_k$ such that $T \geqslant t > s > s_1 > s_2 > \cdots > s_k > 0$. We remark that for any times t, s, s_1 satisfying $t > s > s_1$ the random variables β_t and $\beta_s/s - \beta_{s_1}\big/s_1$ have vanishing covariance, and thus are independent. More generally, for $s > s_1 > s_2 > s_3$ the random variables $\beta_s/s - \beta_{s_1}\big/s_1$ and $\beta_{s_2}\big/s_2 - \beta_{s_3}\big/s_3$ are independent. We note that $\xi_s/s - \xi_{s_1}\big/s_1 = \beta_s/s - \beta_{s_1}\big/s_1$. It follows that

$$\begin{aligned}\mathbb{P}\big(\xi_t \leqslant x|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \xi_{s_k}\big) &= \mathbb{P}\left(\xi_t \leqslant x\,\middle|\,\xi_s, \frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1}, \frac{\xi_{s_1}}{s_1} - \frac{\xi_{s_2}}{s_2}, \ldots, \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k}\right)\\ &= \mathbb{P}\left(\xi_t \leqslant x\,\middle|\,\xi_s, \frac{\beta_s}{s} - \frac{\beta_{s_1}}{s_1}, \frac{\beta_{s_1}}{s_1} - \frac{\beta_{s_2}}{s_2}, \ldots, \frac{\beta_{s_{k-1}}}{s_{k-1}} - \frac{\beta_{s_k}}{s_k}\right).\end{aligned} \tag{206}$$

Since ξ_s and ξ_t are independent of $\beta_s/s - \beta_{s_1}\big/s_1, \beta_{s_1}\big/s_1 - \beta_{s_2}\big/s_2, \ldots, \beta_{s_{k-1}}\big/s_{k-1} - \beta_{s_k}\big/s_k$, the desired result follows immediately. □

Because $\{\xi_t\}$ is a Markov process, the conditional expectation $\mathbb{E}\big[H\big|\mathcal{F}_t^{\xi}\big]$ simplifies to $H_t = \mathbb{E}[H|\xi_t]$ so that we only need to specify the value ξ_t of the process at time t and not the entire trajectory $\{\xi_u\}_{0\leqslant u\leqslant t}$. We shall first establish the following result.

Proposition 11. *Let H be a random variable taking the value E_i with probability π_i $(i = 1, 2, \ldots, n)$, and set $\xi_t = \sigma t H + B_t - (t/T)B_T$ for $0 \leqslant t < T$, where σ is a constant and the Brownian motion $\{B_t\}$ is independent of H. Then the conditional expectation $H_t = \mathbb{E}\big[H\big|\mathcal{F}_t^{\xi}\big]$ is given by*

$$H_t = \frac{\sum_i \pi_i E_i \exp\!\left(\frac{\sigma\xi_t E_i T - \frac{1}{2}\sigma^2 E_i^2 t T}{T-t}\right)}{\sum_i \pi_i \exp\!\left(\frac{\sigma\xi_t E_i T - \frac{1}{2}\sigma^2 E_i^2 t T}{T-t}\right)}. \tag{207}$$

Proof. The conditional expectation $H_t = \mathbb{E}[H|\xi_t]$ can be expressed in terms of the conditional probability as follows:

$$H_t = \sum_i E_i \mathbb{P}(H = E_i|\xi_t). \tag{208}$$

To determine the conditional probability $\mathbb{P}(H = E_i|\xi)$ we note that according to the Bayes formula that we can write

$$\mathbb{P}(H = E_i|\xi_t) = \frac{\pi_i \rho(\xi_t|H = E_i)}{\sum_i \pi_i \rho(\xi_t|H = E_i)}, \tag{209}$$

where

$$\rho(\xi_t|H=E_i)=\sqrt{\tfrac{T}{2\pi t(T-t)}}\exp\left(-\frac{(\xi_t-\sigma t E_i)^2 T}{2t(T-t)}\right). \tag{210}$$

Expression (210) follows from the fact that conditional on $H=E_i$ the variable ξ_t in (201) is normally distributed with mean $\sigma t E_i$ and variance

$$\mathbb{E}[(B_t-(t/T)B_T)^2]=t(T-t)/T. \tag{211}$$

Putting these together, we deduce (207) after some rearrangements of terms. □

From the expression (207) we can infer directly the property that $H_t \to E_k$ as $t\to T$, provided we set $H=E_k$. Writing H_t^k for the conditional energy process $\{H_t(H=E_k)\}$, $\omega_{ij}=E_i-E_j$ for the difference of energy eigenvalues, and substituting $\xi_t=\xi_t^k=\sigma t E_k+\beta_t$ into (207), we obtain

$$\begin{aligned} H_t^k &= \frac{\sum_i \pi_i E_i \exp\left(\frac{\sigma\xi_t^k E_i T-\frac{1}{2}\sigma^2 E_i^2 tT}{T-t}\right)}{\sum_i \pi_i \exp\left(\frac{\sigma\xi_t^k E_i T-\frac{1}{2}\sigma^2 E_i^2 tT}{T-t}\right)} \\ &= \frac{\pi_k E_k+\sum_{i\neq k}\exp\left(\frac{\sigma T\omega_{ik}\beta_t-\frac{1}{2}\sigma^2\omega_{ik}^2 tT}{T-t}\right)}{\pi_k+\sum_{i\neq k}\exp\left(\frac{\sigma T\omega_{ik}\beta_t-\frac{1}{2}\sigma^2\omega_{ik}^2 tT}{T-t}\right)}. \end{aligned} \tag{212}$$

Observe that for each i the numerator in the exponent in (212) approaches a strictly negative number $-\frac{1}{2}\sigma^2\omega_{ik}^2 tT$. Hence, as $t\to T$ all the exponential terms are rapidly suppressed and we are left with the desired outcome: $H_T^k=E_k$.

17. Innovation process for finite-time collapse model

Let us analyse the properties of the process $\{H_t\}$ in (207) more closely. By taking the stochastic differential of (207) we obtain

$$\mathrm{d}H_t=\sigma_t V_t\left[\frac{1}{T-t}(\xi_t-\sigma T H_t)\,\mathrm{d}t+\mathrm{d}\xi_t\right], \tag{213}$$

where $V_t=\mathbb{E}\left[H^2\middle|\mathcal{F}_t^\xi\right]-H_t^2$ is the conditional variance of the random variable H. Clearly, there exists a choice of a process $\{W_t\}$ defined in terms of $\{H_t\}$ and $\{\xi_t\}$ such that the drift term in the dynamical equation (213) can be removed, when expressed in terms of $\{W_t\}$. It remains to be shown that such a process is a Brownian motion that derives the dynamics of the state (198). We shall proceed by verifying the following result.

Proposition 12. *The process* $\{W_t\}$ *defined by*

$$W_t=\int_0^t \frac{1}{T-s}(\xi_s-\sigma T H_s)\,\mathrm{d}s+\xi_t \tag{214}$$

is an $\{\mathcal{F}_t^\xi\}$*-Brownian motion.*

Proof. First we note that the tower property of conditional expectation shows

$$\mathbb{E}\left[B_t\middle|\mathcal{F}_s^\xi\right]=\mathbb{E}\left[\mathbb{E}\left[B_t\middle|\mathcal{F}_t^B,H\right]\middle|\mathcal{F}_s^\xi\right]=\mathbb{E}_s[B_s], \tag{215}$$

where we write $\mathbb{E}_s[-]=\mathbb{E}\left[-\middle|\mathcal{F}_s^\xi\right]$. It follows from (201) that $\xi_t=\mathbb{E}_t[\xi_t]$, and hence that ξ_t is given by

$$\xi_t=\sigma t H_t+\left(1-\frac{t}{T}\right)\mathbb{E}_t[B_t]. \tag{216}$$

We now proceed to establish that $\{W_t\}$ as defined by (214) is an $\{\mathcal{F}_t^\xi\}$-martingale. For $t \leqslant u$, we have

$$\begin{aligned}
\mathbb{E}_t[W_u] &= \mathbb{E}_t[\xi_u] + \mathbb{E}_t\left[\int_0^u \frac{1}{T-s}(\xi_s - \sigma T H_s)\,\mathrm{d}s\right] \\
&= \mathbb{E}_t[\xi_u] + \int_0^t \frac{1}{T-s}(\xi_s - \sigma T H_s)\,\mathrm{d}s + \int_t^u \frac{1}{T-s}(\mathbb{E}_t[\xi_s] - \sigma T H_t)\,\mathrm{d}s.
\end{aligned} \tag{217}$$

Here we have used the fact that $\{\xi_t\}$ and $\{H_t\}$ are $\{\mathcal{F}_t^\xi\}$-adapted and that $\mathbb{E}_t[H_s] = H_t$ for $t \leqslant s$. Therefore,

$$\begin{aligned}
\mathbb{E}_t[W_u] &= \mathbb{E}_t\left[\sigma u H + B_u - \frac{u}{T}B_T\right] + W_t - \xi_t \\
&\quad + \int_t^u \frac{1}{T-s}\mathbb{E}_t\left[\sigma s H + B_s - \frac{s}{T}B_T\right]\mathrm{d}s - \sigma T H_t \int_t^u \frac{1}{T-s}\,\mathrm{d}s \\
&= \sigma u H_t + W_t - \xi_t + \sigma H_t \int_t^u \frac{s}{T-s}\,\mathrm{d}s - \sigma T H_t \int_t^u \frac{1}{T-s}\,\mathrm{d}s \\
&\quad + \mathbb{E}_t[B_t]\left(1 - \frac{u}{T} + \int_t^u \frac{1}{T-s}\left(1 - \frac{s}{T}\right)\mathrm{d}s\right) \\
&= W_t - \xi_t + \sigma t H_t + \mathbb{E}_t[B_t]\left(1 - \frac{t}{T}\right) \\
&= W_t,
\end{aligned} \tag{218}$$

where in the final step we have made use of the relation (216). This establishes the martingale property satisfied by $\{W_t\}$. On the other hand, (214) implies $(\mathrm{d}W_t)^2 = (\mathrm{d}\xi_t)^2$, whereas (201) implies $(\mathrm{d}\xi_t)^2 = \mathrm{d}t$. It follows that $(\mathrm{d}W_t)^2 = \mathrm{d}t$, and this establishes the assertion that $\{W_t\}$ is an $\{\mathcal{F}_t^\xi\}$-Brownian motion. □

We remark that by substituting (214) into (213) we obtain the dynamics

$$\mathrm{d}H_t = \sigma_t V_t\,\mathrm{d}W_t \tag{219}$$

for the process $\{H_t\}$ given in (207), which shows that $\{H_t\}$ is a martingale. On the other hand, by taking the stochastic differential of the energy process $H_t = \langle\psi_t|\hat{H}|\psi_t\rangle/\langle\psi_t|\psi_t\rangle$ using (198) we have obtained (199). To show that (207) is indeed the energy process associated with the dynamics (198) we must demonstrate that the two processes labelled by $\{W_t\}$ are identical. In particular, we have the following result.

Proposition 13. *The innovation process $\{W_t\}$ defined in (214) is the Brownian motion that derives the dynamics of the wavefunction in (198).*

Proof. The stochastic differential equation (198) can be given by the following integral representation:

$$|\psi_t\rangle = \exp\left(-\mathrm{i}\hat{H}t - \frac{1}{4}\int_0^t \sigma_s^2(\hat{H}-H_s)^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \sigma_s(\hat{H}-H_s)\,\mathrm{d}W_s\right)|\psi_0\rangle. \tag{220}$$

This can be expressed more concisely as $|\psi_t\rangle = \hat{U}_t\hat{R}_t|\psi_0\rangle$, where

$$\hat{U}_t = \exp(-\mathrm{i}\hat{H}t) \tag{221}$$

is the usual unitary evolution operator and

$$\hat{R}_t = \exp\left(\frac{1}{2}\int_0^t \sigma_s(\hat{H}-H_s)\,\mathrm{d}W_s - \frac{1}{4}\int_0^t \sigma_s^2(\hat{H}-H_s)^2\,\mathrm{d}s\right) \tag{222}$$

is the 'reduction' operator. The square of $\hat{R}_t$, which we denote by $\hat{M}_t$, is an operator-valued martingale, given by

$$\hat{M}_t = \exp\left(\int_0^t \sigma_s(\hat{H} - H_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2(\hat{H} - H_s)^2\,\mathrm{d}s\right)$$
$$= \frac{\exp\left(\int_0^t \hat{H}\sigma_s(\mathrm{d}W_s + \sigma_s H_s\,\mathrm{d}s) - \frac{1}{2}\int_0^t \sigma_s^2\hat{H}^2\,\mathrm{d}s\right)}{\exp\left(\int_0^t H_s\sigma_s(\mathrm{d}W_s + \sigma_s H_s\,\mathrm{d}s) - \frac{1}{2}\int_0^t \sigma_s^2 H_s^2\,\mathrm{d}s\right)}. \tag{223}$$

Let us now introduce a modified Brownian motion $\{W_t^*\}$ according to

$$W_t^* = W_t + \int_0^t \sigma_s H_s\,\mathrm{d}s, \tag{224}$$

so $\mathrm{d}W_t^* = \mathrm{d}W_t + \sigma_t H_t\,\mathrm{d}t$. While $\{W_t^*\}$ is a drifted Brownian motion in the probability measure $\mathbb{P}$, we can construct another probability measure $\mathbb{Q}$ in which the process $\{W_t^*\}$ becomes a standard Brownian motion. Then, because $\hat{H}$ is constant in time, we can write $\hat{M}_t$ in the simple form

$$\hat{M}_t = \frac{1}{\Phi_t}\exp\left(\hat{H}\int_0^t \sigma_s\,\mathrm{d}W_s^* - \frac{1}{2}\hat{H}^2\int_0^t \sigma_s^2\,\mathrm{d}s\right), \tag{225}$$

where

$$\Phi_t = \exp\left(\int_0^t \sigma_s H_s\,\mathrm{d}W_s^* - \frac{1}{2}\int_0^t \sigma_s^2 H_s^2\,\mathrm{d}s\right) \tag{226}$$

is a positive martingale process.

Recall that (198) preserves the norm of $|\psi_0\rangle$. Therefore, if we assume initially that $\langle\psi_0|\psi_0\rangle = 1$, then it follows that $\langle\psi_0|\hat{M}_t|\psi_0\rangle = 1$ for all t. Thus, we deduce from (223) and (224) that

$$\Phi_t = \langle\psi_0|\exp\left(\hat{H}\int_0^t \sigma_s\,\mathrm{d}W_t^* - \frac{1}{2}\hat{H}^2\int_0^t \sigma_s^2\,\mathrm{d}t\right)|\psi_0\rangle. \tag{227}$$

As a consequence, we can write

$$\hat{M}_t = \frac{\exp\left(\hat{H}\int_0^t \sigma_s\,\mathrm{d}W_t^* - \frac{1}{2}\hat{H}^2\int_0^t \sigma_s^2\,\mathrm{d}t\right)}{\langle\psi_0|\exp\left(\hat{H}\int_0^t \sigma_s\,\mathrm{d}W_t^* - \frac{1}{2}\hat{H}^2\int_0^t \sigma_s^2\,\mathrm{d}t\right)|\psi_0\rangle}, \tag{228}$$

which has the effect of isolating the dependence of $\hat{M}_t$ on $\{H_t\}$. In particular, $\hat{M}_t$ depends on $\{H_t\}$ entirely through the modified Brownian motion $\{W_t^*\}$. The process $\{H_t\}$ in turn is given by $H_t = \langle\psi_t|\hat{H}|\psi_t\rangle/\langle\psi_t|\psi_t\rangle$, from which it follows that $H_t = \langle\psi_0|\hat{H}\hat{M}_t|\psi_0\rangle$. Therefore, by use of (228) we have

$$H_t = \frac{\langle\psi_0|\hat{H}\exp\left(\hat{H}\int_0^t\sigma_s\,\mathrm{d}W_s^* - \frac{1}{2}\hat{H}^2\int_0^t\sigma_s^2\,\mathrm{d}s\right)|\psi_0\rangle}{\langle\psi_0|\exp\left(\hat{H}\int_0^t\sigma_s\,\mathrm{d}W_s^* - \frac{1}{2}\hat{H}^2\int_0^t\sigma_s^2\,\mathrm{d}s\right)|\psi_0\rangle}, \tag{229}$$

which shows that H_t can be expressed in terms of $\{W_t^*\}$ and t. This is given by

$$H_t = \frac{\sum_i \pi_i E_i \exp\left(E_i\int_0^t \sigma_s\,\mathrm{d}W_s^* - \frac{1}{2}E_i^2\int_0^t \sigma_s^2\,\mathrm{d}s\right)}{\sum_i \pi_i \exp\left(E_i\int_0^t \sigma_s\,\mathrm{d}W_s^* - \frac{1}{2}E_i^2\int_0^t \sigma_s^2\,\mathrm{d}s\right)}, \tag{230}$$

where π_i denotes the initial probability that the eigenvalue attained is E_i.

Now if the process $\{H_t\}$ obtained in (207) is the energy process (230), then from the relation $\int_0^t \sigma_s^2\,\mathrm{d}s = \sigma^2 tT/(T-t)$, we deduce, by comparison of (207) and (230), that

$$\xi_t = (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}W_s^* \tag{231}$$

must be satisfied. To show that (231) is satisfied, we remark first that the stochastic differential of (214) is given by

$$\mathrm{d}W_t + \sigma T \frac{1}{T-t} H_t\,\mathrm{d}t = \frac{1}{T-t}\xi_t\,\mathrm{d}t + \mathrm{d}\xi_t. \tag{232}$$

On the other hand, the differential form of (224) is

$$\mathrm{d}W_t + \sigma T \frac{1}{T-t} H_t\,\mathrm{d}t = \mathrm{d}W_t^*. \tag{233}$$

Therefore, by comparing (232) and (233) we deduce the relation

$$\mathrm{d}\xi_t = -\frac{1}{T-t}\xi_t\,\mathrm{d}t + \mathrm{d}W_t^*. \tag{234}$$

This, however, is the differential form of (231). It follows that the process $\{H_t\}$ obtained in (207) is the energy process (230) associated with the collapse model (198). In particular, the process $\{W_t\}$ defined in (214) is the Brownian motion that drives the dynamics of the state (198). □

The above result also shows that the process $\{\xi_t\}$ defined by (201) is itself a Brownian bridge in the $\mathbb{Q}$-measure. This follows from the integral representation (231) above, which shows that in the $\mathbb{Q}$-measure, under which $\{W_t^*\}$ is a standard Brownian motion, $\{\xi_t\}$ is a zero-mean Gaussian process with autocovariance given by

$$\begin{aligned}
\mathbb{E}[\xi_s\xi_t] &= (T-s)(T-t)\mathbb{E}\left[\int_0^s \frac{1}{T-u}\,\mathrm{d}W_u^* \int_0^t \frac{1}{T-v}\,\mathrm{d}W_v^*\right] \\
&= (T-s)(T-t)\left\{\mathbb{E}\left[\left(\int_0^s \frac{1}{T-u}\,\mathrm{d}W_u^*\right)^2\right]\right. \\
&\quad \left. + \mathbb{E}\left[\int_0^s \frac{1}{T-u}\,\mathrm{d}W_u^* \int_s^t \frac{1}{T-u}\,\mathrm{d}W_u^*\right]\right\} \\
&= (T-s)(T-t)\left\{\mathbb{E}\left[\left(\int_0^s \frac{1}{T-u}\,\mathrm{d}W_u^*\right)^2\right]\right. \\
&\quad \left. + \mathbb{E}\left[\int_0^s \frac{1}{T-u}\,\mathrm{d}W_u^*\right]\mathbb{E}\left[\int_s^t \frac{1}{T-u}\,\mathrm{d}W_u^*\right]\right\} \\
&= s\left(1-\frac{t}{T}\right)
\end{aligned} \tag{235}$$

for $s \leqslant t$. Here, we have substituted the integral representation (231) into the right-hand side of (235), applied the Wiener–Ito isometry, and used the independent increments property of Brownian motion. We shall make use of this result to establish Proposition 14.

18. Reverse construction for finite-time collapse model

We have demonstrated in the previous section that the closed-form solution to the stochastic equation (198) can be obtained by use of a nonlinear filtering methodology, in which we have introduced a pair of independent random data H and $\{\beta_t\}$. Conversely, starting from the stochastic equation (198), we can derive the existence of such a pair of independent data.

We let $\{H_t\}$ be the energy process associated with the collapse process (198), and *define* the process $\{\xi_t\}$ in terms of the energy process $\{H_t\}$ and the Brownian motion $\{W_t\}$ according to

$$\xi_t = (T-t)\int_0^t \frac{1}{T-s}(\mathrm{d}W_s + \sigma_s H_s\,\mathrm{d}s). \tag{236}$$

Then we have the following:

Proposition 14. *The random variables H_T and $\beta_t = \xi_t - \sigma t H_T$ are independent for all $t \in [0,T]$. Furthermore, the process $\{\beta_t\}$ is a Brownian bridge.*

Proof. For the independence of the random variables H_T and β_t it suffices to verify that

$$\mathbb{E}[\mathrm{e}^{x\beta_t + yH_T}] = \mathbb{E}[\mathrm{e}^{x\beta_t}]\mathbb{E}[\mathrm{e}^{yH_T}] \tag{237}$$

for arbitrary x, y. Using the tower property of conditional expectation we have

$$\mathbb{E}[\mathrm{e}^{x\beta_t + yH_T}] = \mathbb{E}[\mathrm{e}^{x\xi_t}\mathbb{E}[\mathrm{e}^{(y-\sigma t x)H_T}|\xi_t]]. \tag{238}$$

Let us consider the inner expectation $\mathbb{E}[\mathrm{e}^{(y-\sigma t x)H_T}|\xi_t]$. Using expressions (209) and (210) for the conditional probability distribution of the terminal energy H_T, we deduce that

$$\mathbb{E}[\mathrm{e}^{(y-\sigma t x)H_T}|\xi_t] = \Phi_t^{-1}\sum_i \pi_i\,\mathrm{e}^{(y-\sigma t x)E_i}\exp\left(\frac{\sigma\xi_t E_i T - \frac{1}{2}\sigma^2 E_i^2 tT}{T-t}\right), \tag{239}$$

where the process $\{\Phi_t\}$ is defined in (226). Recall now that $\{\Phi_t\}$ is the density process for changing the measure from $\mathbb{Q}$ to $\mathbb{P}$. As a consequence, we have

$$\mathbb{E}[\mathrm{e}^{x\xi_t}\mathbb{E}[\mathrm{e}^{(y-\sigma t x)H_T}|\xi_t]] = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\xi_t}\sum_i \pi_i\,\mathrm{e}^{(y-\sigma t x)E_i}\exp\left(\frac{\sigma\xi_t E_i T - \frac{1}{2}\sigma^2 E_i^2 tT}{T-t}\right)\right]. \tag{240}$$

However, the process $\{\xi_t\}$ appearing in (240) is a Brownian bridge under the $\mathbb{Q}$-measure. Therefore, the expectation in (240) can be computed by elementary methods and we deduce, after some rearrangement of terms, that

$$\mathbb{E}[\mathrm{e}^{x\beta_t + yH_T}] = \sum_i \pi_i\,\mathrm{e}^{yE_i}\,\mathrm{e}^{\frac{t(T-t)}{2T}x^2}. \tag{241}$$

Here, we have used the facts that if g is a zero-mean Gaussian random variable with variance γ^2, then $\mathbb{E}[\mathrm{e}^{xg}] = \mathrm{e}^{\frac{1}{2}\gamma^2 x^2}$, and that the variance of the $\mathbb{Q}$-Brownian bridge $\{\xi_t\}$ is $t(T-t)/T$. This proves the independence of $\{\beta_t\}$ and H_T. The result (241) also establishes that under $\mathbb{P}$ the process $\{\beta_t\}$ is Gaussian and has mean zero and variance $t(T-t)/T$. To establish $\{\beta_t\}$ is a Brownian bridge, we must show that for $s \leqslant t$ the covariance of β_s and β_t is given by $s(T-t)/T$. Alternatively, we can analyse the moment generating function $\mathbb{E}[\mathrm{e}^{x\beta_s + y\beta_t}]$. We thus proceed as follows. First, using the tower property of conditional expectation we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{x\beta_s + y\beta_t}] &= \mathbb{E}[\mathrm{e}^{x\xi_s + y\xi_t - \sigma(xs+yt)H_T}] \\
&= \mathbb{E}[\mathrm{e}^{x\xi_s + y\xi_t}\mathbb{E}[\mathrm{e}^{-\sigma(xs+yt)H_T}|\xi_t]] \\
&= \mathbb{E}\left[\mathrm{e}^{x\xi_s + y\xi_t}\Phi_t^{-1}\sum_i \pi_i\,\mathrm{e}^{-\sigma(xs+yt)E_i}\exp\left(\frac{\sigma\xi_t E_i T - \frac{1}{2}\sigma^2 E_i^2 tT}{T-t}\right)\right] \\
&= \sum_i \pi_i \exp\left(-\sigma(xs+yt)E_i - \frac{tT}{2(T-t)}\sigma^2 E_i^2\right) \\
&\quad \times \mathbb{E}^{\mathbb{Q}}\left[\exp\left(x\xi_s + \left(y + \frac{\sigma T E_i}{T-t}\right)\xi_t\right)\right].
\end{aligned} \tag{242}$$

Now we note that

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(x\xi_s + \left(y + \frac{\sigma T E_i}{T-t}\right)\xi_t\right)\right] = \exp\left\{\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\left(x\xi_s + \left(y + \frac{\sigma E_i T}{T-t}\right)\xi_t\right)^2\right]\right\}$$
$$= \exp\left\{\frac{1}{2}\left(x^2 s\left(1-\frac{s}{T}\right) + \left(y + \frac{\sigma E_i T}{T-t}\right)^2 t\left(1-\frac{t}{T}\right)\right.\right.$$
$$\left.\left. + 2x\left(y + \frac{\sigma E_i T}{T-t}\right) s\left(1-\frac{t}{T}\right)\right)\right\}, \tag{243}$$

from which we observe that the dependence on the energy eigenvalues $\{E_i\}$ in the summand of (242) drops out. As a consequence, we obtain

$$\mathbb{E}[\mathrm{e}^{x\beta_s + y\beta_t}] = \exp\left\{\frac{1}{2}\left(x^2 s\left(1-\frac{s}{T}\right) + y^2 t\left(1-\frac{t}{T}\right) + 2xys\left(1-\frac{t}{T}\right)\right)\right\}. \tag{244}$$

It follows that the covariance of β_s and β_t for $s \leqslant t$ is given by

$$\frac{\partial^2}{\partial x \partial y}\mathbb{E}[\mathrm{e}^{x\beta_s + y\beta_t}]\bigg|_{x=y=0} = s\left(1-\frac{t}{T}\right). \tag{245}$$

This establishes the assertion that $\{\beta_t\}$ is a $\mathbb{P}$-Brownian bridge. □

19. Time-change and Brownian bridge

The asymptotic collapse model and the finite-time collapse model that have been investigated in this paper are in fact related by an elementary time-change. In this section we shall demonstrate how a finite-time collapse model can be seen to 'emerge' from an asymptotic collapse model, and vice verse, by the use of time-change techniques applied to Brownian motion.

We begin by noting the following property of Brownian motion. Suppose $f(s) > 0$ is a continuous monotonic function over $s \in [0, T]$ such that

$$\int_0^t f^2(s)\,\mathrm{d}s \to \infty \tag{246}$$

as $t \to T$, where $0 < T \leqslant \infty$. Let $\tau(t)$ be given by the solution of the equation

$$\int_0^{\tau(t)} f^2(s)\,\mathrm{d}s = t, \tag{247}$$

or equivalently,

$$f^2(\tau(t)) = \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^{-1}, \tag{248}$$

and let $\{B_t\}$ be a standard Brownian motion. Then the process $\{X_t\}$ defined by

$$X_t = \int_0^{\tau(t)} f(s)\,\mathrm{d}B_s \tag{249}$$

is a standard Brownian motion. To verify that $\{X_t\}$ is a Brownian motion it suffices to show that the covariance of X_s and X_t for $s \leqslant t$ is given by s. This follows on account of the fact that since $\tau(t)$ is deterministic, (249) shows that $\{X_t\}$ is Gaussian with mean zero. Using Ito's lemma we then find that the covariance is given by

$$\mathbb{E}[X_s X_t] = \int_0^{\tau(s)} f^2(s)\,\mathrm{d}s \tag{250}$$

for $s \leqslant t$. Therefore, from the defining relation (247) we conclude that the covariance of X_s and X_t is indeed s. As an example, consider the function

$$f(s) = \frac{T}{T - s}. \tag{251}$$

Clearly $f(s)$ is monotonic and satisfies $f(s) \to \infty$ as $s \to T$. Furthermore, we have

$$\int_0^\tau f^2(s)\,\mathrm{d}s = \frac{\tau T}{T - \tau}. \tag{252}$$

Substitution of (251) into (247) shows that the relevant time-change in this example is

$$\tau(t) = \frac{tT}{t + T}. \tag{253}$$

As a consequence, we find that the process $\{\tilde{B}_t\}$ defined by

$$\tilde{B}_t = \int_0^{\frac{tT}{t+T}} \frac{T}{T - s}\,\mathrm{d}B_s \tag{254}$$

is a standard Brownian motion. From (253), we find that $\tau(0) = 0$ and that $\tau(t) \to T$ as $t \to \infty$. Therefore, the time-change (253) has the effect of 'slowing down' the process.

We can consider, conversely, a time-change that has the effect of 'speeding up' the process. For this we require the following variant of the previous result. Suppose $f(s) > 0$ is a continuous monotonic function over $s \in [0, \infty]$ such that

$$\int_0^t f^2(s)\,\mathrm{d}s \to T \tag{255}$$

as $t \to \infty$, where $0 < T < \infty$. Let $\tau(t)$ be given by the solution of equation (247), and let $\{B_t\}$ be a standard Brownian motion. Then the process $\{Y_t\}$ defined by

$$Y_t = \int_0^{\tau(t)} f(s)\,\mathrm{d}B_s \tag{256}$$

is a standard Brownian motion. This result can be verified by studying the covariance of Y_s and Y_t. As an example we consider the function

$$f(s) = \frac{T}{s + T}, \tag{257}$$

which clearly satisfies the condition (255). It follows that the relevant time-change in this example is

$$\tau(t) = \frac{tT}{T - t}, \tag{258}$$

or equivalently,

$$t = \frac{\tau T}{\tau + T}, \tag{259}$$

and that the process $\{Y_t\}$ defined by

$$Y_t = \int_0^{\frac{tT}{T-t}} \frac{T}{s + T}\,\mathrm{d}B_s \tag{260}$$

is a standard Brownian motion. We find from (258) that $\tau(0) = 0$ and that $\tau(t) \to \infty$ as $t \to T$. Therefore, in this example the time-change has the effect of speeding up the clock variable $\tau(t)$.

With these results at hand we now proceed to establish the relationship between the finite-time and the asymptotic collapse models studied in this paper. To begin we recall the definition

$$\xi_t = \sigma t H + \beta_t \tag{261}$$

for the information process $\{\xi_t\}$ in the finite-time collapse model, and the fact that the Brownian bridge process $\{\beta_t\}$ admits the integral representation

$$\beta_t = (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}B_s. \tag{262}$$

Next, we consider the time-change given by (258) and define $\{\eta_\tau\}$ by

$$\eta_\tau = \frac{1}{T-t}\xi_t. \tag{263}$$

Then substituting (261) here and using the integral representation (262) we obtain

$$\eta_\tau = \frac{\sigma H t}{T-t} + \frac{\beta_t}{T-t} = \sigma H\tau + \int_0^t \frac{1}{T-s}\,\mathrm{d}B_s, \tag{264}$$

on account of the integral representation (262). However, recalling the relations (259) and (254), we deduce that $\{\eta_\tau\}$ can be expressed in the form

$$\eta_\tau = \sigma H\tau + \tilde{B}_\tau. \tag{265}$$

This defines a 'standard' filtering problem associated with the asymptotic collapse model, if we regard τ as the time variable. In particular, the best estimate for H, given the observation $\{\mathcal{F}^\eta_\tau\}$, is determined by the conditional expectation

$$\tilde{H}_\tau = \mathbb{E}[H|\eta_\tau]. \tag{266}$$

Clearly we have the relation $\tilde{H}_\tau = H_{t(\tau)}$. Furthermore, we have

$$\mathrm{d}\tilde{H}_\tau = \sigma V_\tau\,\mathrm{d}\tilde{W}_\tau, \tag{267}$$

where

$$\tilde{W}_\tau = \eta_\tau - \sigma\int_0^\tau \tilde{H}_s\,\mathrm{d}s \tag{268}$$

is a standard Brownian motion. From (263) and (268) we deduce that

$$\mathrm{d}\left(\frac{1}{T-t}\xi_t\right) = \sigma\tilde{H}_\tau\,\mathrm{d}\tau + \mathrm{d}\tilde{W}_\tau = \sigma H_t\frac{1}{(T-t)^2}\,\mathrm{d}t + \frac{1}{T-t}\,\mathrm{d}W_t, \tag{269}$$

where

$$\tilde{W}_{\frac{tT}{T-t}} = \int_0^t \frac{1}{T-s}\,\mathrm{d}W_s. \tag{270}$$

Expanding the left-hand side of (269) we deduce

$$\mathrm{d}\xi_t + \frac{1}{T-t}(\xi_t - \sigma H_t)\,\mathrm{d}t = \mathrm{d}W_t, \tag{271}$$

which, if integrated, reduces to the relation (214). In this manner we find that by taking the asymptotic collapse model (265) and applying the time-change according to (258), which has the effect of 'speeding up' the process, we recover the finite-time collapse model (261).

Conversely, given the model

$$\xi_t = \sigma t H + \tilde{B}_t \tag{272}$$

that solves the asymptotic collapse model, we may consider the time-change given by (253) and define

$$\eta_\tau = \frac{T}{t+T}\xi_t = \sigma\tau H + \frac{T}{t+T}\tilde{B}_t. \tag{273}$$

However, because of the relations $t = \tau T/(T-\tau)$ and (254) we deduce that

$$\frac{T}{t+T}\tilde{B}_t = (T-\tau)\int_0^\tau \frac{1}{T-s}\,\mathrm{d}B_s, \tag{274}$$

which is the integral representation for the Brownian bridge process $\{\beta_\tau\}$ with respect to the time variable τ. As a consequence, we obtain

$$\eta_\tau = \sigma\tau H + \beta_\tau, \tag{275}$$

and we thus recover the model that solves the finite-time collapse process. We thus obtain the following conclusion:

Proposition 15. *The finite-time collapse model (198) can be obtained from the asymptotic collapse model (3) by means of the time-change defined in (258). The reverse transformation is obtained by the time-change defined in (253).*

20. Discussion

The result of Proposition 9 demonstrates that given the dynamical equation (3) for the quantum state and the associated energy process (4) we can deduce the existence of the asymptotic random variable H_∞ and an independent noise process $\{B_t\}$. The variable H_∞ carries the interpretation of a *hidden variable* in the stochastic quantum theory. More precisely, because H_∞ is $\{\mathcal{F}^W_\infty\}$-measurable, its value can only be determined with certainty after the collapse has taken place. The 'quantum noise' process $\{B_t\}$ represents the 'disinformation' that hides H_∞ before the completion of the collapse process.

Whether the energy-based reduction models considered here suffice to describe measurements and relaxation phenomena in general in nonrelativistic quantum mechanics remains an open issue. There are attempts, for example, to formulate a spontaneous collapse of the wave packet in a localized region in space (see, e.g., [8], for a recent work in this area). However, localization of a particle in a small region in space typically requires large energy. Indeed, in a generic measurement-theoretic context one requires an infinite amount of energy to confine a particle in a finite region [9], and hence it may be unphysical to speak of a true 'position measurement'. Quantities such as the position or momentum of a particle thus represent what an experimentalist can *estimate* from appropriate energy measurements.

This observation is consistent with the point of view put forward by Wiener [31] that "under the quantum mechanics, it is impossible to obtain any information giving the position or momentum of a particle, much less the two together, without a positive effect on the energy of the particle examined . . .Thus all coupling is strictly a coupling involving energy. . ." The basic idea is that, in order to measure a physical quantity of a system the measurement apparatus must interact with the system, and this is achieved in the form of interchange of particles (typically photons or phonons). When these particles interact with the measurement apparatus they create some form of excitation which then allows the device to estimate quantities of interest. In this regard, we can take the point of view that the energy-based models are of fundamental importance in describing random phenomena involving quantum systems.

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